Answer to #78163, Physics/Electromagnetism

Solution:

When a number of progressive waves of slightly different wavelength in a group superpose each other, the velocity with which the wave packet or point of reinforcement advances in the medium is called group velocity.

Superposition of number of progressive waves of slightly different wavelength constitutes wave group

Consider two wave of different frequencies ω_1 and ω_2 are travelling in same direction and the amplitude of the wave are same

$$\psi_1 = A\cos(\omega_1 t - k_1 x)$$

$$\psi_2 = A\cos(\omega_2 t - k_2 x)$$

On superposition of two waves

$$\psi = \psi_1 + \psi_2 = A\cos(\omega_1 t - k_1 x) + A\cos(\omega_2 t - k_2 x)$$

$$\psi = 2A\cos\left(\frac{\omega_1 t - k_1 x + \omega_2 t - k_2 x}{2}\right)\cos\left(\frac{\omega_1 t - k_1 x - \omega_2 t + k_2 x}{2}\right)$$

$$\psi = 2A\cos\left[\left(\frac{\omega_1 - \omega_2}{2}\right)t - \left(\frac{k_1 - k_2}{2}\right)x\right]\cos\left[\left(\frac{\omega_1 + \omega_2}{2}\right)t - \left(\frac{k_1 + k_2}{2}\right)x\right]$$

$$\psi = 2A\cos[\Delta\omega \times t - \Delta k \times x]\cos[\omega \times t - k \times x]$$

$$\psi = A_m \cos[\omega \times t - k \times x]$$

$$A_m = 2A\cos(t\Delta\omega - x\Delta k)$$

A_m is the modulated amplitude

$$A_m = 2A\cos\Delta k\left(t\frac{\Delta\omega}{\Delta k} - x\right)$$

$\frac{\Delta\omega}{\Delta k}$ is the group velocity(V_g)

If the difference in the frequencies of two waves of the group is small, then

$$V_g = \frac{d\omega}{dk}$$

Relation between group velocity and phase velocity

The phase velocity of the wave is given by

$$V_p = \frac{\omega}{k}$$

$$\omega = kV_p$$

$$V_g = \frac{d\omega}{dk}$$

$$V_g = \frac{d(kV_p)}{dk}$$

$$V_g = V_p + k\frac{dV_p}{dk}$$

but

$$k = \frac{2\pi}{\lambda}$$

So

$$V_g = V_p + \frac{2\pi}{\lambda}\frac{dV_p}{dk}$$

$$V_g = V_p + \frac{2\pi}{\lambda}\frac{dV_p}{d\lambda}\times\frac{d\lambda}{dk}$$

$$V_g = V_p + \frac{2\pi}{\lambda}\frac{dV_p}{d\lambda}\times\frac{-\lambda^2}{2\pi}$$

$$V_g = V_p - \lambda\frac{dV_p}{d\lambda}$$

This show the relationship between V_g and V_p in dispersive medium