Answer on question # 77788 - Physics / Mechanics | Relativity

1. A string fixed at both ends (x = 0 and x = l) starts to oscillate under a suddenly applied distributed load with constant density q. Find the vibrational pattern if at the initial moment the string was at rest.

Solution.

1) String equation

$$\frac{\partial^2 U(x,y)}{\partial t^2} = c^2 \frac{\partial^2 U(x,y)}{\partial x^2} + q$$

a. Border conditions

$$U(0,t) = 0; U(l,t) = 0$$

b. Initial conditions

$$U(x,0) = 0; U(x',0) = 0$$

c. Search for a solution in the form

$$U(x,y) = v(x,y) + w(x)$$

2) Search for a solution w(x)

a. Equation

$$c^2 \frac{\partial^2 w(x)}{\partial x^2} + q = 0$$

i. Border conditions

$$w(0) = 0; w(l) = 0$$

b. Equation with separable variables

$$\partial^2 w = -\frac{q}{c^2} \partial x^2$$

c. Integration

$$w(x) = -\frac{qx^2}{2c^2} + C_1 x + C_2$$

d. Find the integration constants using the initial conditions

$$C_1 = 0;\ C_2 = \frac{ql}{2c^2}$$

e. Solution from above

$$w(x) = \frac{qx(l-x)}{2c^2}$$

3) Search for a solution v(x)

a. Equation for v(x)

$$\frac{\partial^2 v(x,t)}{\partial t^2} - c^2 \frac{\partial^2 v(x,t)}{\partial x^2} = 0$$

i. Border conditions

$$v(0,t) = 0; v(l,t) = 0$$

ii. Initial conditions

$$v(x,0) = -w(x);\ v(x',0) = 0$$

b. Search for a solution in the form

$$v(x,t) = T(t)X(x)$$

c. Substitute this solution in the equation

$$T''(t)X(x) = c^2 T(t)X''(x)$$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Equality exists if this relation does not depend on t or x. Then these relations are equal to some constant λ

d. We have two differential equation

$$T''(t) + \lambda c^2 T(t) = 0$$
$$X''(x) + \lambda X(x) = 0$$

e. Solution for X(x)

   i. We choose solutions for X(x) at λ>0, since other cases (λ=0, λ<0) make a trivial solution.

   ii. General solution

$$X(x) = Acos(\sqrt{\lambda}x) + Bsin(\sqrt{\lambda}x)$$

   iii. Using border conditions

$$X(0) = 0; X(l) = 0$$

   iv. We have the eigenfunctions and eigenvalues

$$A = 0; B = 1$$
$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 ; \; X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$$

f. Solution for T(t)

   i. We choose solutions for T(t) at λ>0, since other cases (λ=0, λ<0) make a trivial solution.

   ii. General solution

$$T_n(t) = A_n cos\left(\frac{n\pi ct}{l}\right) + B_n sin\left(\frac{n\pi ct}{l}\right)$$

g. General solution is, written as a linear combination of basic solutions

$$v(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

h. Using the initial conditions, we find A_n, B_n

$$A_n = \frac{2}{l}\int_0^l v(x,0)\sin\left(\frac{n\pi x}{l}\right)dx$$
$$B_n = \frac{2}{\lambda_n l}\int_0^l v'(x,0)\sin\left(\frac{n\pi x}{l}\right)dx$$
$$A_n = \frac{2}{l}\int_0^l \frac{qx}{2c^2}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$
$$B_n = \frac{2}{\lambda_n l}\int_0^l 0 * \sin\left(\frac{n\pi x}{l}\right)dx = 0$$

i. Integration

$$A_n = \frac{2}{l}\int_0^l \frac{qx}{2c^2}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{q}{lc^2}\left[\int_0^l xl\sin\left(\frac{n\pi x}{l}\right)dx - \int_0^l x^2\sin\left(\frac{n\pi x}{l}\right)dx\right]$$
$$A_n = \frac{2ql^2}{c^2(n\pi)^3}[\cos(n\pi - 1)]$$

j. Not zero only for odd

$$A_n = \frac{4ql^2}{c^2\left((2n+1)\pi\right)^3}$$

k. Private solution

$$v(x,t) = \frac{4ql^2}{c^2\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\cos\left(\frac{(2n+1)\pi ct}{l}\right)\sin\left(\frac{(2n+1)\pi x}{l}\right)$$

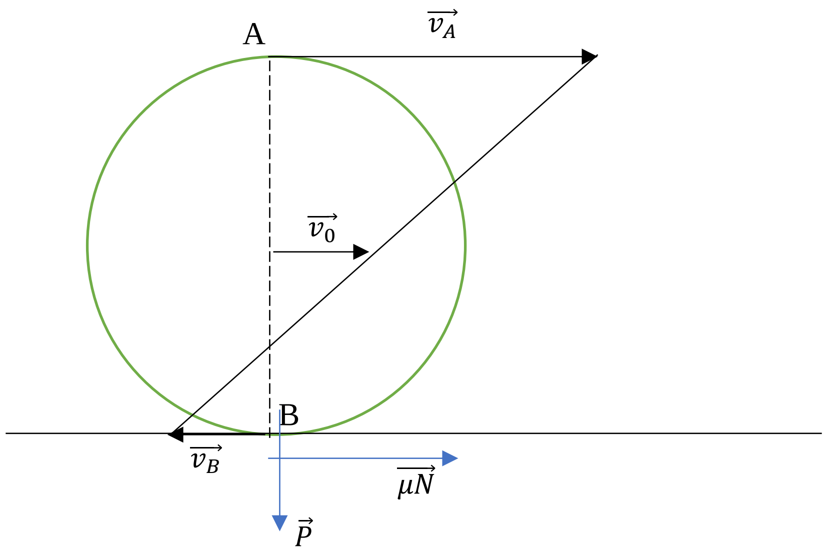
4) We finally have a solution

$$U(x,t) = \frac{4ql^2}{c^2\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\cos\left(\frac{(2n+1)\pi ct}{l}\right)\sin\left(\frac{(2n+1)\pi x}{l}\right) + \frac{qx(l-x)}{2c^2}$$

2. A uniform solid disk with mass M and radius R is placed on a horizontal plane at time t = 0. The sliding and rolling friction coefficients are, respectively, μ и fk. Initial velocity of the center of mass is v0 and angular velocity is ω0. Find the times t1 and t2, at which the slipping finishes and the disk stops respectively

Solution.

1) Consider the movement of the disk when it slides



a. Make the law of rotational motion

$$\frac{d\omega}{dt} = \frac{\mu M g R}{J}$$

$J = \frac{MR^2}{2}$ —moment of inertia.

b. Equation for angular speed

$$\omega = \omega_0 - \frac{2\mu g}{R} t$$

c. When the disk stop slides we have angular speed

$$\omega_1 = \omega_0 - \frac{2\mu g}{R} t_1$$

d. During braking, the center of mass will gain speed

$$v_C = v_0 + \mu g t_1$$

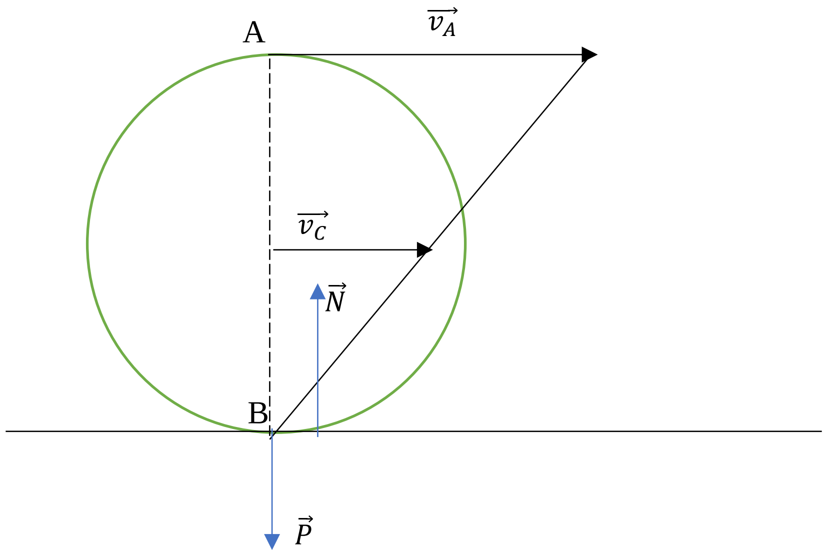
e. And disk will gain angular speed

$$\omega_1 = \frac{v_0 + \mu g t_1}{R}$$

f. However, time for stop slides

$$t_1 = \frac{\omega_0 R - v_0}{3\mu g}$$

2) Consider the movement of the disk when it stop slides



a. In this case, only the rolling friction force acts. Make the law of rotational motion

$$\frac{fMg}{J} = \frac{d\omega}{dt}$$

$J = \frac{MR^2}{2}$ – moment of inertia

b. Equation of angular velocity

$$\omega = \omega_1 - \frac{2fg}{R^2}t$$

c. When disk stopped

$$\omega = 0$$

$$0 = \frac{v_0 + \mu g t_1}{R} - \frac{2fg}{R^2}t_2$$

$$\frac{v_0 + \mu g\left(\frac{\omega_0 R - v_0}{3\mu g}\right)}{R} = \frac{2fg}{R^2}t_2$$

$$t_2 = \frac{(2v_0 + \omega_0 R)R}{6fg}$$

Answer

Time, when disk stop sliding

$$t_1 = \frac{\omega_0 R - v_0}{3\mu g}$$

Time, when disk stopped

$$t_2 = \frac{(2v_0 + \omega_0 R)R}{6fg}$$

All time

$$t = t_1 + t_2 = \frac{\omega_0 R - v_0}{3\mu g} + \frac{(2v_0 + \omega_0 R)R}{6fg}$$