A bead of mass m slides without friction on a vertical hoop of radius R. The bead moves under the combined action of gravity and a spring, with spring constant k , attached to the bottom of the hoop. Assume that the equilibrium (relaxed) length of the spring is R. The bead is released from rest at θ = 0 with a non-zero but negligible speed to the right.

(a) What is the speed v of the bead when θ = 90∘? Express your answer in terms of m, R, k, and g.
(b) What is the magnitude of the force the hoop exerts on the bead when θ = 90∘ ? Express your answer in terms of m, R, k, and g.

Solution

As the equilibrium (relaxed) length of the spring is R the force due to the spring is $k(r-R)$, where r is the length of the spring.

At the top of the hoop the gravitational potential energy of the bead is $mg(2R)$ and potential energy due to the spring is $\frac{1}{2}k(2R-R)^2=\frac{1}{2}kR^2$.

Hence the initial potential energy is

$$U_i=\frac{1}{2}kR^2+2mgR.$$

Since all the forces are conservative, the mechanical energy is constant and we have

$$K_i+U_i=K_f+U_f.$$

The initial kinetic energy is zero and we obtain

$$K_f=U_i-U_f.$$

When θ = 90∘ the gravitational potential energy of the bead is mgR. The length of the spring is $\sqrt{R^2+R^2}=\sqrt{2}R$. Potential energy due to the spring is $\frac{1}{2}k\left(\sqrt{2}R-R\right)^2=\frac{1}{2}kR\left(\sqrt{2}-1\right)^2$.

Hence the final potential energy is

$$U_f=\frac{1}{2}kR^2\left(\sqrt{2}-1\right)^2+mgR.$$

We have

$$\frac{mv_f^2}{2}=\left(\frac{1}{2}kR^2+2mgR\right)-\left(\frac{1}{2}kR^2\left(\sqrt{2}-1\right)^2+mgR\right).$$

The speed of the bead when θ = 90∘

$$v_f=\sqrt{\frac{2\left(\sqrt{2}-1\right)kR^2}{m}+2gR}$$

We can apply the second Newton’s law to the bead when θ = 90∘:

$$\overrightarrow{ma} = \overrightarrow{mg} + \overrightarrow{k(\sqrt{2}R - R)} + \vec{N}.$$

Consider radial projection of it:

$$0 = k\left(\sqrt{2}R - R\right)\cos 45° - N.$$

The magnitude of the force the hoop exerts on the bead when θ = 90∘

$$N = k\left(\sqrt{2}R - R\right)\cos 45° = kR\left(\sqrt{2} - 1\right)\frac{\sqrt{2}}{2} = kR\left(1 - \frac{1}{\sqrt{2}}\right).$$

Answer: (a) $\sqrt{\frac{2(\sqrt{2}-1)kR^2}{m} + 2gR}$**; (b)** $kR\left(1 - \frac{1}{\sqrt{2}}\right)$**.**