Answer on Question #86336 – Math – Differential Equations

Question

Solve the following ODE using the power series method:

(x^2-1)y''+3xy'+xy=0

Solution

We seek the solution of the equation in the form of a power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ *Then* $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ *and* $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$:

$$(x^2-1)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + 3x\sum_{n=1}^{\infty} a_n n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + 3\sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + 3\sum_{n=1}^{\infty} a_n n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n - 2a_2 - 6a_3 x - \sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1)x^n + 3a_1 x + 3\sum_{n=2}^{\infty} a_n n x^n + a_0 x + \sum_{n=2}^{\infty} a_{n-1} x^n = 0$$

$$\begin{cases} -2a_2 = 0 \\ -6a_3 + 3a_1 + a_0 = 0 \\ a_n n(n-1) - a_{n+2}(n+2)(n+1) + 3a_n n + a_{n-1} = 0, \qquad n = 2,3,4\ldots \end{cases}$$

$$\begin{cases} a_2 = 0 \\ a_3 = \dfrac{3a_1 + a_0}{6} \\ a_n n(n+2) - a_{n+2}(n+2)(n+1) + a_{n-1} = 0, \qquad n = 2,3,4\ldots \end{cases}$$

$$\begin{cases} a_2 = 0 \\ a_3 = \dfrac{3a_1 + a_0}{6} \\ a_{n+2} = \dfrac{a_n n(n+2) + a_{n-1}}{(n+2)(n+1)}, \qquad n = 2,3,4\ldots \end{cases}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \frac{3a_1 + a_0}{6}x^3 + \sum_{n=2}^{\infty} a_{n+2} x^{n+2}$$
$$= a_0 + a_1 x + \frac{3a_1 + a_0}{6}x^3 + \sum_{n=2}^{\infty} \frac{a_n n(n+2) + a_{n-1}}{(n+2)(n+1)} x^{n+2}$$

Answer: $y(x) = a_0 + a_1 x + \frac{3a_1 + a_0}{6} x^3 + \sum_{n=2}^{\infty} \frac{a_n n(n+2) + a_{n-1}}{(n+2)(n+1)} x^{n+2}$.