ANSWER on Question #86334 – Math – Differential Equations

QUESTION

Show that, for the differential equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0$$

e^{mx} is a particular integral if $m^2 + am + b = 0$. Hence find the value of m so that e^{mx} is a particular integral of the equation

$$(x-2)\frac{d^2y}{dx^2} - (4x-7)\frac{dy}{dx} + \ (4x-6)y = 0$$

SOLUTION

Find the indicated derivatives of the function $y = e^{mx}$ and substitute in the initial equation

$$y = e^{mx} \to \begin{cases} \dfrac{dy}{dx} = \dfrac{d}{dx}(e^{mx}) = m \cdot e^{mx} \\ \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(m \cdot e^{mx}) = m^2 \cdot e^{mx} \end{cases}$$

Then,

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0 \to m^2 \cdot e^{mx} + a(x) \cdot m \cdot e^{mx} + b(x) \cdot e^{mx} = 0 \to$$

$$e^{mx} \cdot \left(m^2 + a(x)m + b(x)\right) = 0 \to \begin{cases} e^{mx} \neq 0, & \forall x \in \mathbb{R}, \forall m \in \mathbb{R} \\ m^2 + a(x)m + b(x) = 0 \end{cases}$$

Conclusion,

$$\boxed{\begin{array}{c} if\ m\ is\ solution\ of\ m^2 + a(x)m + b(x) = 0, then \\ y = e^{mx}\ is\ partial\ integral\ of\ \dfrac{d^2y}{dx^2} + a(x)\dfrac{dy}{dx} + b(x)y = 0 \end{array}}$$

Hint: 1) There is an error in the condition, since the $a(x)$ and $b(x)$ record implies that $a(x)$ and $b(x)$ are some functions of variable x, the form of which is not specified. A note $m^2 + am + b = 0$ suggests that a and b in arbitrary constants.

2) This equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0$$

and the substitution $y = e^{mx}$ suggests the Euler-Cauchy equation. But there, it is assumed that $a(x) = a_0 x^n$ and $b(x) = b_0 x^k$, to come to a similar equation $m^2 + am + b = 0$.

(More information: https://en.wikipedia.org/wiki/Cauchy%E2%80%93Euler_equation)

If this is not done, then in general, the equation $m^2 + a(x)m + b(x) = 0$ is an equation with two variables, which cannot be solved in the literal sense of the word, but can only be used to express the variable m through the variable x.

This idea will clearly demonstrate the next part of the task.

$$(x-2)\frac{d^2y}{dx^2} - (4x-7)\frac{dy}{dx} + (4x-6)y = 0$$

The solution of the equation will be sought as $y = e^{mx}$. Find the indicated derivatives of the function $y = e^{mx}$ and substitute in the initial equation

$$y = e^{mx} \rightarrow \begin{cases} \dfrac{dy}{dx} = \dfrac{d}{dx}(e^{mx}) = m \cdot e^{mx} \\ \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(m \cdot e^{mx}) = m^2 \cdot e^{mx} \end{cases}$$

Then,

$$(x-2)\frac{d^2y}{dx^2} - (4x-7)\frac{dy}{dx} + (4x-6)y = 0 \rightarrow$$

$$(x-2)m^2 \cdot e^{mx} - (4x-7)m \cdot e^{mx} + (4x-6) \cdot e^{mx} = 0 \rightarrow$$

$$e^{mx} \cdot \big((x-2)m^2 - (4x-7)m + (4x-6)\big) = 0 \rightarrow$$

$$\begin{cases} e^{mx} \neq 0, \qquad \forall x \in \mathbb{R}, \forall m \in \mathbb{R} \\ (x-2)m^2 - (4x-7)m + (4x-6) = 0 \end{cases}$$

Got an equation with two variables x and m:

$$(x-2)m^2-(4x-7)m+\ (4x-6)=0$$

This is a quadratic equation for the variable m. Let's try to solve it using the discriminant formula

$$\begin{cases} a=x-2 \\ b=-(4x-7) \\ c=4x-6 \end{cases} \to D=b^2-4ac=\left(-(4x-7)\right)^2-4\cdot(x-2)\cdot(4x-6)\to$$

$$D=16x^2-56x+49-4\cdot(4x^2-6x-8x+12)\to$$

$$D=16x^2-56x+49-16x^2+24x+32x-48=1\to\sqrt{D}=\sqrt{1}=1$$

$$\left[\begin{aligned} m_1&=\frac{-b-\sqrt{D}}{2a}=\frac{4x-7-1}{2(x-2)}=\frac{4x-8}{2(x-2)}=\frac{2(2x-4)}{2(x-2)}=\frac{2x-4}{x-2} \\ m_1&=\frac{-b+\sqrt{D}}{2a}=\frac{4x-7+1}{2(x-2)}=\frac{4x-6}{2(x-2)}=\frac{2(2x-3)}{2(x-2)}=\frac{2x-3}{x-2} \end{aligned}\right.$$

As we see, m is a function of the variable x, not a number.

Hint: I can only indicate that you solve the specified equation on the site https://www.wolframalpha.com

That can get a solution

$$y(x)=\frac{C_1e^{2x-4}\sqrt{x-2}}{\sqrt{4-2x}}+\frac{C_2e^{2x-4}(x-4)\sqrt{x-2}x}{\sqrt{2}\sqrt{2-x}}$$

(https://www.wolframalpha.com/input/?i=(x-2)++(d%5E2+y)%2F(dx%5E2+)-(4x-7)++dy%2Fdx%2B+(4x-6)y%3D0)

We can see that in a machine solution there is a degree e^{2x-4}.

Q.E.D.