Answer to Question #86295 - Math - Calculus

Question:

Show that for two scalar fields f and g : $\nabla.\left(\nabla f\times\left(f\nabla g\right)\right)=0$.

Solution:

We know that $\nabla.\left(\vec{A}\times\vec{B}\right)=\left(\nabla\times\vec{A}\right).\vec{B}-\left(\nabla\times\vec{B}\right).\vec{A}$

$\nabla.\left(\nabla f\times\left(f\nabla g\right)\right)=\left(\nabla\times\nabla f\right).\left(f\nabla g\right)-\left(\nabla\times\left(f\nabla g\right)\right).\nabla f$

For any scalar field φ, $\nabla\times\nabla\varphi=0$.

So $\nabla\times\nabla f=0$

Also $\nabla\times\left(f\nabla g\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ f\dfrac{\partial g}{\partial x} & f\dfrac{\partial g}{\partial y} & f\dfrac{\partial g}{\partial z}\end{vmatrix}$

$$=\left[\vec{i}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)-\vec{j}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right)+\vec{k}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\right]$$

Now $\left(\nabla\times\left(f\nabla g\right)\right).\nabla f=\left[\vec{i}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)-\vec{j}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right)+\vec{k}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\right].\left[\vec{i}\frac{\partial f}{\partial x}+\vec{j}\frac{\partial f}{\partial y}+\vec{k}\frac{\partial f}{\partial z}\right]$

$$=\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)\frac{\partial f}{\partial x}-\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right)\frac{\partial f}{\partial y}+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\frac{\partial f}{\partial z}$$

$$=\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial x}\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}+\frac{\partial f}{\partial y}\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial x}\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}=0$$

Hence $\left(\nabla\times\left(f\nabla g\right)\right).\nabla f=0$.

Thus $\nabla.\left(\nabla f\times\left(f\nabla g\right)\right)=0$.