ANSWER on Question #86028 – Math – Algebra

QUESTION

Use Weierstrass' inequalities to prove that

$$\left(\sum_{i=1}^{n}\frac{1}{\sqrt{i}}\right)\le\frac{1}{\sqrt{n!}}\cdot\prod_{i=2}^{n}\sqrt{i-1}+2\cdot\left(\sum_{i=2}^{n}\frac{1}{\sqrt{i}}\right)$$

SOLUTION

In In mathematics, the Weierstrass product inequality states that,

For given real numbers $0\le a_1,a_2,a_3,\dots,a_n\le 1$

$$(1-a_1)(1-a_2)(1-a_3)\cdots(1-a_n)\ge 1-S_n\left(where\ S_n=\sum_{i=1}^{n}a_i\right)\leftrightarrow\prod_{i=1}^{n}(1-a_i)\ge 1-\sum_{i=1}^{n}a_i$$

$$(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_n)\ge 1+S_n\left(where\ S_n=\sum_{i=1}^{n}a_i\right)\leftrightarrow\prod_{i=1}^{n}(1+a_i)\ge 1+\sum_{i=1}^{n}a_i$$

(More information: https://en.wikipedia.org/wiki/Weierstrass_product_inequality)

Or more generally

$$\prod_{i}(1-x_i)^{\omega_i}\ge 1-\sum_{i}\omega_i x_i\,,\ \ where\ x_i\le 1,\ \ and\ either\ \omega_i\ge 1(for\ all\ i)\ or\ \omega_i\le 0(for\ all\ i)$$

In our case,

$$\left(\sum_{i=1}^{n}\frac{1}{\sqrt{i}}\right)\le\frac{1}{\sqrt{n!}}\cdot\prod_{i=2}^{n}\sqrt{i-1}+2\cdot\left(\sum_{i=2}^{n}\frac{1}{\sqrt{i}}\right)\to$$

$$\frac{1}{\sqrt{1}}+\left(\sum_{i=2}^{n}\frac{1}{\sqrt{i}}\right)-2\cdot\left(\sum_{i=2}^{n}\frac{1}{\sqrt{i}}\right)\le\frac{1}{\sqrt{n!}}\cdot\prod_{i=2}^{n}\sqrt{i\cdot\left(1-\frac{1}{i}\right)}\to$$

$$1-\left(\sum_{i=2}^{n}\frac{1}{\sqrt{i}}\right)\le\frac{1}{\sqrt{n!}}\cdot\prod_{i=2}^{n}\sqrt{i}\cdot\left(1-\frac{1}{i}\right)^{\frac{1}{2}}$$

Transform the right side of the inequality

$$\frac{1}{\sqrt{n!}}\cdot\prod_{i=2}^{n}\sqrt{i}\cdot\left(1-\frac{1}{i}\right)^{\frac{1}{2}}\equiv\frac{1}{\sqrt{n!}}\cdot\left(\prod_{i=2}^{n}\sqrt{i}\right)\cdot\left(\prod_{i=2}^{n}\left(1-\frac{1}{i}\right)^{\frac{1}{2}}\right)=\frac{\overbrace{\sqrt{1\cdot 2\cdot 3\cdot 4\cdots n}}^{\sqrt{n!}}}{\sqrt{n!}}\cdot\left(\prod_{i=2}^{n}\left(1-\frac{1}{i}\right)^{1/2}\right)=$$

$$=\frac{\sqrt{n!}}{\sqrt{n!}}\cdot\left(\prod_{i=2}^{n}\left(1-\frac{1}{i}\right)^{\frac{1}{2}}\right)=\left(\prod_{i=2}^{n}\left(1-\frac{1}{i}\right)^{1/2}\right)$$

Conclusion,

$$\frac{1}{\sqrt{n!}} \cdot \prod_{i=2}^{n} \sqrt{i} \cdot \left(1 - \frac{1}{i}\right)^{\frac{1}{2}} \equiv \left(\prod_{i=2}^{n} \left(1 - \frac{1}{i}\right)^{1/2}\right)$$

Then, it remains to prove

$$1 - \left(\sum_{i=2}^{n} \frac{1}{\sqrt{i}}\right) \leq \prod_{i=2}^{n} \left(1 - \frac{1}{i}\right)^{1/2}$$

Since,

$$1 - \left(\sum_{i=2}^{n} \frac{1}{2} \cdot \frac{1}{i}\right) \leq \prod_{i=2}^{n} \left(1 - \frac{1}{i}\right)^{\frac{1}{2}} - on\ Weierstrass\ inequality$$

And

$$2n \geq \sqrt{n}, \ \ \forall n \geq 2 \rightarrow \frac{1}{2n} \leq \frac{1}{\sqrt{n}}, \ \ \forall n \geq 2 \rightarrow \sum_{i=2}^{n} \frac{1}{2} \cdot \frac{1}{i} \leq \sum_{i=2}^{n} \frac{1}{\sqrt{i}} \rightarrow$$

$$1 - \sum_{i=2}^{n} \frac{1}{\sqrt{i}} \leq 1 - \sum_{i=2}^{n} \frac{1}{2} \cdot \frac{1}{i}$$

Then,

$$1 - \sum_{i=2}^{n} \frac{1}{\sqrt{i}} \leq 1 - \left(\sum_{i=2}^{n} \frac{1}{2} \cdot \frac{1}{i}\right) \leq \prod_{i=2}^{n} \left(1 - \frac{1}{i}\right)^{\frac{1}{2}} \rightarrow$$

$$1 - \left(\sum_{i=2}^{n} \frac{1}{\sqrt{i}}\right) \leq \prod_{i=2}^{n} \left(1 - \frac{1}{i}\right)^{1/2}$$

Q.E.D.