Answer on Question #86005 – Math – Differential Equations

Question

$p^2 + q^2 - 2px - 2qy + 2xy$ by Charpit's method find the complete integral this differential equation.

Solution

Charpit's method is:

$$\frac{\mathrm{dp}}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \qquad (1)$$

and

$$dz = pdx + qdy \quad , \qquad (2)$$

where

$$f(x, y, z, p, q) = p^2 + q^2 - 2px - 2qy + 2xy \, , \qquad (3)$$

$$\frac{\partial f}{\partial x} = -2p + 2y, \frac{\partial f}{\partial y} = -2q + 2x, \frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial p} = 2p - 2x, \frac{\partial f}{\partial q} = 2q - 2y. \quad (4)$$

Let’s substitute (4) in (1):

$\frac{\mathrm{dp}}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dz}{-p(2p-2x)-q(2q-2y)} = \frac{dx}{2x-2p} = \frac{dy}{2y-2q}$, then take such fractions:

1) $\frac{\mathrm{dp}}{-2p+2y} = \frac{dq}{-2q+2x}$,

$\int \frac{\mathrm{dp}}{-2p+2y} = \int \frac{dq}{-2q+2x}$,

$\ln|p - y| = \ln|q - x| + \ln c_1$,

$p - y = c_1(q - x)$,

$p = y + c_1(x - q)$,

$$p = y + c_1 x - c_1 q. \qquad (5)$$

2) $\frac{dx}{2x-2p} = \frac{dy}{2y-2q}$,

$\int \frac{dx}{2x-2p} = \int \frac{dy}{2y-2q}$,

$\ln|x - p| = \ln|y - q| + \ln c_2$,

$x - p = c_2(y - q)$,

$$p = x - c_2 y + c_2 q. \qquad (6)$$

Let’s substitute (5) in (6):

$y + c_1 x - c_1 q = x - c_2 y + c_2 q,$

$q = \frac{(1+c_2)y+(c_1-1)x}{c_1+c_2}.$ (7)

Let's substitute (7) in (5):

$p = y + c_1 x - \frac{y+c_2y+c_1x-x}{c_1+c_2},$

$p = \frac{(c_1+c_2)y+c_1(c_1+c_2)x-(1+c_2)y-(c_1-1)x}{c_1+c_2},$

$p = \frac{(c_1-1)y+\left(c_1(c_1+c_2)-(c_1-1)\right)x}{c_1+c_2}.$ (8)

Let's substitute (7), (8) in (2):

$dz = \frac{(c_1-1)y+\left(c_1(c_1+c_2)-(c_1-1)\right)x}{c_1+c_2}dx + \frac{(1+c_2)y+(c_1-1)x}{c_1+c_2}dy,$

$\int dz = \int \frac{(c_1-1)y+\left(c_1(c_1+c_2)-(c_1-1)\right)x}{c_1+c_2}dx + \frac{(1+c_2)y+(c_1-1)x}{c_1+c_2}dy,$

$z = \frac{(c_1-1)yx+\left(c_1(c_1+c_2)-(c_1-1)\right)x^2}{c_1+c_2} + \frac{(1+c_2)y^2+(c_1-1)xy}{c_1+c_2} + C,$

$z = \frac{(c_1-1)yx+\left(c_1(c_1+c_2)-(c_1-1)\right)x^2+(1+c_2)y^2+(c_1-1)xy}{c_1+c_2} + C,$

$z = \frac{\left(c_1(c_1+c_2)-(c_1-1)\right)x^2+(1+c_2)y^2+2(c_1-1)yx}{c_1+c_2} + C.$

Answer: $z = \frac{\left(c_1(c_1+c_2)-(c_1-1)\right)x^2+(1+c_2)y^2+2(c_1-1)yx}{c_1+c_2} + C$ is the complete integral of the differential equation $p^2 + q^2 - 2px - 2qy + 2xy.$