**Answer on Question #85923 – Math – Statistics and Probability**

**Question**

Let $X_1, X_2, \dots, X_n$ be a random sample from a Poisson distribution with parameter $\lambda$. Find an estimator of $\lambda$ using
**(i)** the method of moments;
**(ii)** the method of maximum likelihood.
Also, compare the estimators obtained in parts i) and ii).

**Solution**

**(i)** $MME$
We know that $E(X) = \lambda$, from which we have a moment estimator of $\lambda$ as

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Also, because we have $Var(X) = \lambda$, equating the second moments, we can see that $\lambda = E(X^2) - (E(X))^2$ so that

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$$

Both are moment estimators of $\lambda$. Thus, the moment estimators may not be unique. We generally choose $\overline{X}$ as an estimator of $\lambda$, for its simplicity.

**(ii)** $MLE$
We have the probability mass function

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0, 1, 2, \dots, \quad \lambda > 0$$

Hence, the likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}$$

Taking the natural logarithm, we have

$$\ln L(\lambda) = \sum_{i=1}^{n} x_i \ \ln \lambda - n\lambda - \sum_{i=1}^{n} \ln(x_i!)$$

Differentiate both sides with respect to $\lambda$

$$\frac{d}{d\lambda}\big(L(\lambda)\big) = \frac{1}{\lambda}\sum_{i=1}^{n} x_i - n$$

Find the value of $\lambda$ which maximizes $L(\lambda)$

$$\frac{d}{d\lambda}\big(L(\lambda)\big) = 0 => \frac{1}{\lambda}\sum_{i=1}^{n} x_i - n = 0$$

That is

$$\lambda = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$$

The second derivative

$$\frac{d^2}{d\lambda^2}\big(L(\lambda)\big) = -\frac{1}{\lambda^2}\sum_{i=1}^{n} x_i < 0 \text{ for all } \lambda$$

Therefore, $MLE$ of $\lambda$ is

$$\hat{\lambda} = \overline{X},$$

which is the same as Method of Moments estimator.