Answer on Question #85825 – Math – Calculus

Question

Using Green's Theorem evaluate the integral

$\int x^2 y dx - xy^2\, dy$ (1)

where C is the circle $x^2 + y^2$ =4 oriented counter clockwise.

Solution

Green's Theorem:

$\oint Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$. (2)

Use (2) for (1)

P=$x^2 y$, Q=$-xy^2$, $\frac{\partial Q}{\partial x} = -y^2$, $\frac{\partial P}{\partial y} = x^2$, $I = \int x^2 y dx - xy^2\, dy = \iint_D (-\, y^2 - x^2) dxdy.$

Let's move in the integral I to polar coordinates:

$x = rcos\, \varphi, y = rsin\, \varphi$, then

$$\mathrm{I} = \int_0^{2\pi} d\varphi \int_0^{2} (-r^2 sin^2\varphi - r^2 cos^2\, \varphi) r dr = \int_0^{2\pi} d\varphi \int_0^{2} (-r^3 (sin^2\varphi + cos^2\, \varphi)) dr =$$

$$= \int_0^{2\pi} d\varphi \int_0^{r} (-r^3) dr = \int_0^{2\pi} \left(-\frac{r^4}{4}\right)\Big|_{r=0}^{r=2} d\varphi = \int_0^{2\pi} (-4) d\varphi = -4\varphi\Big|_{\varphi=0}^{\varphi=2\pi} = -8\pi.$$

Answer: $\int x^2 y dx - xy^2\, dy = -8\pi.$