## ANSWER on Question #85725 – Math – Differential Equations

## QUESTION

Consider the differential equation

$$x^3y''' + 10x^2y'' + 16xy' - 16y = 0 \to x, x^{-4}, x^{-4}\ln(x), \quad \forall x \in (0,\infty)$$

Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval.

The functions satisfy the differential equation and are linearly independent since

$$W(x, x^{-4}, x^{-4}\ln(x)) \neq 0, \qquad for\ 0 < x < \infty$$

Form the general solution.

$$y_{general}(x) = \cdots$$

## SOLUTION

**Definition**: for $n$ real- or complex-valued functions $\{f_1, f_2, \dots, f_n\}$ which are $(n-1)$ times differentiable on an interval $I$, the Wronskian $W(f_1, f_2, \dots, f_n)$ as a function on $I$ is defined by

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

( More information: https://en.wikipedia.org/wiki/Wronskian )

In our case,

$$W(x, x^{-4}, x^{-4}\ln(x)) \to \begin{cases} f_1(x) = x \\ f_2(x) = x^{-4} \\ f_3(x) = x^{-4}\ln(x) \end{cases} \to \begin{cases} f_1'(x) = 1 \\ f_2'(x) = -4 \cdot x^{-5} \\ f_3'(x) = -4 \cdot x^{-5}\ln(x) + x^{-4} \cdot x^{-1} \end{cases} \to$$

$$\begin{cases} f_1'(x) = 1 \\ f_2'(x) = -4 \cdot x^{-5} \\ f_3'(x) = x^{-5} \cdot (1 - 4\ln(x)) \end{cases} \to \begin{cases} f_1''(x) = 0 \\ f_2''(x) = (-4)(-5) \cdot x^{-6} \\ f_3''(x) = -5 \cdot x^{-6} \cdot (1 - 4\ln(x)) + x^{-5} \cdot (-4x^{-1}) \end{cases} \to$$

$$\begin{cases} f_1''(x) = 0 \\ f_2''(x) = 20 \cdot x^{-6} \\ f_3''(x) = x^{-6}(-5 + 20\ln(x) - 4) \end{cases} \to \begin{cases} f_1''(x) = 0 \\ f_2''(x) = 20 \cdot x^{-6} \\ f_3''(x) = x^{-6}(-9 + 20\ln(x)) \end{cases}$$

Conclusion,

$$\boxed{\begin{cases} f_1(x) = x \\ f_2(x) = \dfrac{1}{x^4} \\ f_3(x) = \dfrac{\ln(x)}{x^4} \end{cases} \to \begin{cases} f_1'(x) = 1 \\ f_2'(x) = -\dfrac{4}{x^5} \\ f_3'(x) = \dfrac{1 - 4\ln(x)}{x^5} \end{cases} \to \begin{cases} f_1''(x) = 0 \\ f_2''(x) = \dfrac{20}{x^6} \\ f_3''(x) = \dfrac{-9 + 20\ln(x)}{x^6} \end{cases}}$$

Then,

$$W(x, x^{-4}, x^{-4}\ln(x)) = \begin{vmatrix} x & \dfrac{1}{x^4} & \dfrac{\ln(x)}{x^4} \\ 1 & -\dfrac{4}{x^5} & \dfrac{1 - 4\ln(x)}{x^5} \\ 0 & \dfrac{20}{x^6} & \dfrac{-9 + 20\ln(x)}{x^6} \end{vmatrix} =$$

$$= x \cdot \left(-\frac{4}{x^5}\right) \cdot \frac{-9 + 20\ln(x)}{x^6} + 1 \cdot \frac{20}{x^6} \cdot \frac{\ln(x)}{x^4} + 0 \cdot \frac{1}{x^4} \cdot \frac{1 - 4\ln(x)}{x^5} -$$

$$-0 \cdot \left(-\frac{4}{x^5}\right) \cdot \frac{\ln(x)}{x^4} - x \cdot \frac{20}{x^6} \cdot \frac{1 - 4\ln(x)}{x^5} - 1 \cdot \frac{1}{x^4} \cdot \frac{-9 + 20\ln(x)}{x^6} =$$

$$= \frac{36 - 80\ln(x)}{x^{10}} + \frac{20\ln(x)}{x^{10}} + -0 - 0 - \frac{20 - 80\ln(x)}{x^{10}} - \frac{20\ln(x) - 9}{x^{10}} =$$

$$= \frac{36 - 80\ln(x) + 20\ln(x) - (20 - 80\ln(x)) - (20\ln(x) - 9)}{x^{10}} =$$

$$= \frac{36 - 80\ln(x) + 20\ln(x) - 20 + 80\ln(x) - 20\ln(x) + 9}{x^{10}} =$$

$$= \frac{(36 - 20 + 9) + \ln(x) \cdot (-80 + 20 + 80 - 20)}{x^{10}} = \frac{25}{x^{10}}$$

Conclusion,

$$\boxed{W(x, x^{-4}, x^{-4}\ln(x)) = \frac{25}{x^{10}} \neq 0 \quad for \quad 0 < x < \infty \to \{x, x^{-4}, x^{-4}\ln(x)\} - linearly\ independence}$$

Then,

$$y_{general}(x) = C_1 \cdot x + C_2 \cdot x^{-4} + C_3 \cdot x^{-4}\ln(x) + C_4$$

For verification, we can use

https://www.wolframalpha.com/input/?i=x%5E3*y%27%27%27%2B10*x%5E2*y%27%27%2B16*x*y%27-16y%3D0

**ANSWER**

$$W(x, x^{-4}, x^{-4}\ln(x)) = \frac{25}{x^{10}} \neq 0 \quad for \quad 0 < x < \infty \rightarrow \{x, x^{-4}, x^{-4}\ln(x)\} - linearly\ independence$$

$$y_{general}(x) = C_1 \cdot x + C_2 \cdot x^{-4} + C_3 \cdot x^{-4}\ln(x) + C_4$$