Answer on Question #85484 – Math – Calculus

Question

Obtain the reduction formula for &int (cos x)^n dx,
; where n is an integer which is greater
than 1. Hence evaluate ∫
π/ 2
0
(cos x)^6 dx .

Solution

We have

$$\int \cos^n x\, dx$$

Rewrite the integral in the form

$$\int \cos^{n-1} x \cdot \cos x\, dx$$

Use the formula for integration by parts

$$\int u dv = uv - \int v du$$

Let

$$u = \cos^{n-1} x$$

$$dv = \cos x\, dx$$

Then

$$du = (\cos^{n-1} x)' dx = (n-1)\cos^{n-2} x \cdot (-\sin x)\, dx$$

$$v = \sin x$$

So integration by parts gives

$$\int \cos^n x\, dx = \sin x \cdot \cos^{n-1} x - \int (n-1)\cos^{n-2} x \cdot (-\sin x) \sin x dx$$

or

$$\int \cos^n x\, dx = \sin x \cdot \cos^{n-1} x + (n-1) \int \cos^{n-2} x \cdot \sin^2 x dx$$

Since $\sin^2 x = 1 - \cos^2 x$, we have

$$\int \cos^n x\, dx = \sin x \cdot \cos^{n-1} x + (n-1) \int \cos^{n-2} x \cdot (1 - \cos^2 x) dx$$

or

$$\int \cos^n x\, dx = \sin x \cdot \cos^{n-1} x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

This can be regarded as an equation to be solved for the unknown integral. Adding

$$(n-1) \int \cos^n x dx$$

to both sides, we obtain

$$n\int \cos^n x\,dx = \sin x\cdot\cos^{n-1}x + (n-1)\int \cos^{n-2}x dx$$

or

$$\int \cos^n x\,dx = \frac{1}{n}\sin x\cdot\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x dx$$

This is desired reduction formula.

Now we evaluate

$$\int_0^{\pi/6} \cos^6 x\,dx$$

Using the reduction formula for $n = 6$ we get

$$\int_0^{\pi/2} \cos^6 x\,dx = \frac{1}{6}\sin x\cdot\cos^5 x|_0^{\pi/2} + \frac{5}{6}\int_0^{\pi/2} \cos^4 x\,dx = 0 + \frac{5}{6}\int_0^{\pi/2} \cos^4 x\,dx$$

Now we use the reduction formula for $n = 4$

$$\frac{5}{6}\int_0^{\pi/2} \cos^4 x\,dx = \frac{5}{6}\cdot\left(\frac{1}{4}\sin x\cdot\cos^3 x|_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2} \cos^2 x\,dx\right) = 0 + \frac{5}{8}\int_0^{\pi/2} \cos^2 x\,dx$$

Finally for $n = 2$ we get

$$\frac{5}{8}\int_0^{\pi/2} \cos^2 x\,dx = \frac{5}{8}\cdot\left(\frac{1}{2}\sin x\cdot\cos x|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \cos^0 x\,dx\right) = 0 + \frac{5}{16}\int_0^{\pi/2} dx$$

$$\frac{5}{16}\int_0^{\pi/2} dx = \frac{5}{16}x|_0^{\pi/2} = \frac{5}{16}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$$

Answer: the reduction formula is

$\int \cos^n x\,dx = \frac{1}{n}\sin x\cdot\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x dx,$

$$\int_0^{\pi/2} \cos^6 x\,dx = \frac{5\pi}{32}$$