Answer on Question #85446 – Math – Statistics and Probability

Question

Let X be a discrete random variable having a pmf $f(x)$ given by

$$f(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, 2, \dots \\ 0, & otherwise \end{cases}$$

Show that $f(x)$ is probability distribution and find the expected value (mean) of X.

Solution

$$P(X = x) = p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \dots, \lambda > 0$$

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} > 0, \quad x = 0, 1, 2, \dots$$

$$p(x) = 0, otherwise$$

$$F(k) = \sum_{x=0}^{k} \frac{e^{-\lambda}\lambda^x}{x!}, \quad k = 0, 1, 2, \dots$$

$$If\ k_2 > k_1, then F(k_2) = \sum_{x=0}^{k_2} \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{k_1} \frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=k_1+1}^{k_2} \frac{e^{-\lambda}\lambda^x}{x!} > \sum_{x=0}^{k_1} \frac{e^{-\lambda}\lambda^x}{x!},$$

$k_1 = 0, 1, 2, \dots, k_2 = 1, 2, \dots$

$F(k)$ is an non − decreasing function.

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda}\left(e^{\lambda}\right) = 1$$

Therefore, $f(x)$ is probability distribution.

Find the expected value (mean) of X.

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = 0 + \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} = e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} =$$

$$= e^{-\lambda}\lambda \sum_{t=0}^{\infty} \frac{\lambda^t}{t!} = e^{-\lambda}\lambda\left(e^{\lambda}\right) = \lambda.$$