Answer on Question #85443 – Math – Calculus

Question

If

$y = a\cos\ln x + b\sin\ln x,$ (1)

Show that

$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$ (2)

($y^{(n)}$ means nth derivative of y).

Solution

To prove this, we use the method of mathematical induction:

1) Check, when n=0,

$x^2y^{(2)} + xy^{(1)} + y = 0$? (3)

$y^{(1)} = \frac{-a\sin\ln x}{x} + \frac{b\cos\ln x}{x} = \frac{-1}{x}(a\sin\ln x - b\cos\ln x),$ (4)

$y^{(2)} = \frac{1}{x^2}(a\sin\ln x - b\cos\ln x) + \frac{-1}{x^2}(a\cos\ln x + b\sin\ln x)\,.$ (5)

Substitute (1), (4), (5) in (3):

$x^2y^{(2)} + xy^{(1)} + y = \cancel{a\sin\ln x} - \cancel{b\cos\ln x} - \cancel{a\cos\ln x} - \cancel{b\sin\ln x} - \cancel{a\sin\ln x} + \cancel{b\cos\ln x} + \cancel{a\cos\ln x} + \cancel{b\sin\ln x} = 0\,.$

2) Assume that with n=m (m=1, 2, 3, 4,…) executed (2) for (1)

$x^2y^{(m+2)} + (2m+1)xy^{(m+1)} + (m^2+1)y^{(m)} = 0\,.$ (6)

3) Let's prove with n=m+1 (m=1, 2, 3, 4, …) (2) holds for the function from (1). For this we differentiate (6):

$(x^2y^{(m+2)} + (2m+1)xy^{(m+1)} + (m^2+1)y^{(m)})^{(1)} = (x^2y^{(m+2)})^{(1)} + ((2m+1)xy^{(m+1)})^{(1)} +$

$+((m^2+1)y^{(m)})^{(1)} = 2xy^{(m+2)} + x^2y^{(m+2+1)} + (2m+1)y^{(m+1)} + (2m+1)xy^{(m+1+1)} + (m^2+$
$+1)y^{(m+1)} = x^2y^{(m+2+1)} + (2m+2+1)xy^{(m+1+1)} + (m^2+2m+1+1)\,y^{(m+1)} = x^2y^{(m+1+2)} +$

$+(2(m+1)+1)xy^{(m+1+1)} + ((m+1)^2+1)y^{(m+1)} = x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0.$

If $y = a\cos\ln x + \sin\ln x$, then

$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$ ($y^{(n)}$ means nth derivative of y) for n=0,1,2,3,… .