Answer on Question #84919 – Math – Statistics and Probability

Question

The probability that a certain plant will die within x hours in a certain environment is estimated to be $[1-(1+x^2)^{-1}]$. Determine the probabilities that the plant will die within 2 hours and that it will survive more than 3 hours. Find the corresponding density function.

Solution

Let X be a continuous random variable. The cumulative distribution function (CDF), or briefly the distribution function, for a random variable X is defined by

$$F(x) = P(X \le x) = \int_{-\infty}^{\infty} f(x)\,dx$$

We have that

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - \dfrac{1}{1+x^2}, & x \ge 0 \end{cases}$$

If $x_2 > x_1 > 0$, then $F(x_2) = 1 - \dfrac{1}{1+(x_2)^2} > 1 - \dfrac{1}{1+(x_1)^2} = F(x_1)$

The CDF is non-decreasing.

$F(0) = 1 - \frac{1}{1+(0)^2} = 0,$

$\lim\limits_{x\to 0^+} F(x) = \lim\limits_{x\to 0^+}\left(1 - \frac{1}{1+x^2}\right) = 1 - \frac{1}{1+(0)^2} = 0,$

$\lim\limits_{x\to \infty} F(x) = \lim\limits_{x\to \infty}\left(1 - \frac{1}{1+x^2}\right) = 1 - 0 = 1.$

The probability that the plant will die within 2 hours is equal to

$$P(X \le 2) = F(2) - F(0) = 1 - \frac{1}{1+(2)^2} - \left(1 - \frac{1}{1+(0)^2}\right) = \frac{4}{5} = 0.8$$

The probability that the plant will survive more than 3 hours is equal to

$$P(X > 3) = 1 - P(X \le 3) = 1 - \left(1 - \frac{1}{1+(3)^2}\right) = \frac{1}{10} = 0.1$$

The function $f(x)$ is the so- called density function (PDF) if

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

The cumulative distribution function (CDF)

$$F(x) = P(X \le x) = \int_{-\infty}^{\infty} f(x)\,dx$$

Then

$$f(x) = F'(x)$$

We have that

$$F(x)=\begin{cases}0, & x<0\\ 1-\dfrac{1}{1+x^2}, & x\geq 0\end{cases}$$

$$\left(1-\frac{1}{1+x^2}\right)'=-\left(-\frac{1}{(1+x^2)^2}\right)(2x)=\frac{2x}{(1+x^2)^2}$$

Thus, the corresponding density function is

$$f(x)=\begin{cases}0, & x<0\\ \dfrac{2x}{(1+x^2)^2}, & x\geq 0\end{cases}$$

Answer:

$P(X\leq 2)=\frac{4}{5}=0.8,\quad P(X>3)=\frac{1}{10}=0.1,\quad f(x)=\begin{cases}0, & x<0,\\ \frac{2x}{(1+x^2)^2}, & x\geq 0.\end{cases}$