Answer to Question #84710 - Math - Calculus

Question:

What is the limit of $\left(\frac{e^x-1}{x}\right)^{1/x}$ as x tends to zero?

Solution:

By exponent property, $a^x = e^{\ln a^x} = e^{x\ln a}$.

Using the above property,

$$\left(\frac{e^x-1}{x}\right)^{1/x} = e^{\frac{1}{x}\ln\left(\frac{e^x-1}{x}\right)}$$

$$\lim_{x\to 0} e^{f(x)} = e^{\lim_{x\to 0} f(x)}$$

$$\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{e^x-1}{x}\right) = \lim_{x\to 0}\frac{\ln\left(\frac{e^x-1}{x}\right)}{x} = \lim_{x\to 0}\frac{\frac{x}{e^x-1}\cdot\frac{xe^x-e^x+1}{x^2}}{1}$$ By L'Hospital rule.

$$= \lim_{x\to 0}\frac{xe^x-e^x+1}{x\left(e^x-1\right)} = \lim_{x\to 0}\frac{xe^x+e^x-e^x}{xe^x+e^x-1}$$ By L'Hospital rule.

$$= \lim_{x\to 0}\frac{xe^x}{xe^x+e^x-1} = \lim_{x\to 0}\frac{xe^x+e^x}{xe^x+e^x+e^x}$$ By L'Hospital rule.

$$= \lim_{x\to 0}\frac{(x+1)e^x}{(x+2)e^x} = \lim_{x\to 0}\frac{(x+1)}{(x+2)} = \frac{1}{2}.$$

$$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{e^x-1}{x}\right)} = e^{1/2} = \sqrt{e}.$$

Thus $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{1/x} = \sqrt{e}$.