Answer on Question #84642 – Math – Linear Algebra

Question

How to obtain the eigenvalues and eigenvectors of the matrix: $M = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution

In such problems, we first find the eigenvalues of the matrix.

Finding eigenvalues

To do this, we find the values of λ which satisfy the characteristic equation of the matrix M, namely those values of λ for which
$\det(M - \lambda I) = 0$
where I is the 3×3 identity matrix

$$\boldsymbol{I} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Form the matrix $M - \lambda I$:

$$M - \lambda I = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

Calculate $\det(M - \lambda I)$

$$\det(M - \lambda I) = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)((2-\lambda)^2 - 3^2) =$$

$$(1-\lambda)((2-\lambda)^2 - 3^2) = (1-\lambda)(2-\lambda-3)(2-\lambda+3) = -(1-\lambda)(1+\lambda)(5-\lambda)$$

The solutions of the equation $\det(M - \lambda I) = -(1-\lambda)(1+\lambda)(5-\lambda) = 0$ are

$$\lambda_1 = -1, \;\; \lambda_2 = 1, \;\; \lambda_1 = 5$$

Finding eigenvectors

We can find the eigenvectors by Gaussian Elimination.

STEP 1: For each eigenvalue λ, we have $(M - \lambda I)X = 0$ where X is the eigenvector associated with eigenvalue λ

STEP 2: Find X by Gaussian elimination. That is, convert the augmented matrix.

$$(M - \lambda I, 0)$$

to row echelon form and solve the resulting linear system by back substitution.

Case 1: $\lambda = -1$. We have to find vectors Xwhich satisfy $(M - \lambda I)X = 0$.

First, form the matrix $M - (-1) \cdot I = M + I$

$$M + I = \begin{pmatrix} 2+1 & 3 & 0 \\ 3 & 2+1 & 0 \\ 0 & 0 & 1+1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Construct the augmented matrix $(M - \lambda I, 0)$and convert it to row echelon form

$$\left(\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Rewriting this augmented matrix as a linear system gives

$$\begin{cases} x_1 + x_2 = 0, \\ x_3 = 0 \end{cases}$$

So the form of eigenvector X_1 is given by:

$$X_1 = \begin{pmatrix} x_1 \\ -x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

for any real number $C_1 \neq 0$

Case 2: $\lambda = 1$

$$M - I = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Construct the augmented matrix $(M - \lambda I, 0)$and convert it to row echelon form

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

Rewriting this augmented matrix as a linear system gives

$$\begin{cases} x_1 = 0, \\ x_2 = 0 \\ x_3 = x_3 \end{cases}$$

So the form of eigenvector X_2 is given by:

$$X_2 = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = C_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for any real number $C_2 \neq 0$.

Case 3: $\lambda = 5$.

$$M - 5I = \begin{pmatrix} 2-5 & 3 & 0 \\ 3 & 2-5 & 0 \\ 0 & 0 & 1-5 \end{pmatrix} = \begin{pmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

Construct the augmented matrix $(M - \lambda I, 0)$and convert it to row echelon form

$$\left(\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Rewriting this augmented matrix as a linear system gives

$$\begin{cases} x_1 - x_2 = 0, \\ x_3 = 0 \end{cases} => \begin{cases} x_1 = x_2, \\ x_3 = 0 \end{cases}$$

So the form of eigenvector X_3 is given by:

$$X_3 = \begin{pmatrix} x_1 \\ x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = C_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

for any real number $C_3 \neq 0$.

Answer: Eigenvalues of M are $\lambda_1 = -1, \;\; \lambda_2 = 1, \;\; \lambda_1 = 5$, eigenvectors of M are $X_1 = C_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, X_2 = C_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, X_3 = C_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, where $C_1 \neq 0, C_2 \neq 0, C_3 \neq 0$.