Answer on Question #82459 – Math – Analytic Geometry

Question

Find the equations of those tangent planes to the sphere

$$x^2 + y^2 + z^2 + 2x - 4y + 6z - 7 = 0$$

which intersect in the line $6x - 3y - 23 = 0, 3z + 2 = 0$.

Solution

The equation of the sphere

$x^2 + y^2 + z^2 + 2x - 4y + 6z - 7 = 0$

$(x^2 + 2x + 1) + (y^2 - 4y + 4) + (z^2 + 6z + 9) - 1 - 4 - 9 - 7 = 0$

$(x + 1)^2 + (y - 2)^2 + (z + 3)^2 = 21 \qquad (1)$

Its centre is $(-1, 2, -3)$ and radius $= \sqrt{21}$.

The equation of the lines

$$6x - 3y - 23 = 0, \quad 3z + 2 = 0$$

Any plane through this line is

$(6x - 3y - 23) + k(3z + 2) = 0$

or

$6x - 3y + 3kz + 2k - 23 = 0 \qquad (2)$

It will touch the sphere (1) if perpendicular from centre on plane (2) = radius of sphere (1)

$$\frac{|6(-1) - 3(2) + 3k(-3) + 2k - 23|}{\sqrt{(6)^2 + (-3)^2 + (3k)^2}} = \sqrt{21}$$

$$|7k + 35| = \sqrt{21}\sqrt{45 + 9k^2}$$

$$49(k + 5)^2 = 21(45 + 9k^2)$$

$$7k^2 + 70k + 175 = 135 + 27k^2$$

$$20k^2 - 70k - 40 = 0$$

$$2k^2 - 7k - 4 = 0$$

$k = \dfrac{7 \pm \sqrt{(-7)^2 - 4(2)(-4)}}{2(2)} = \dfrac{7 \pm 9}{4}$

$k_1 = \dfrac{7 - 9}{4} = -\dfrac{1}{2}$

$k_2 = \dfrac{7 + 9}{4} = 4$

Putting these values of k in (2), we get

$6x - 3y + 3\left(-\dfrac{1}{2}\right)z + 2\left(-\dfrac{1}{2}\right) - 23 = 0$

$6x - 3y - \dfrac{3}{2}z - 24 = 0$

$4x - 2y - z - 16 = 0$

$6x - 3y + 3(4)z + 2(4) - 23 = 0$

$2x - y + 4z - 5 = 0$

Answer: $4x - 2y - z - 16 = 0$ and $2x - y + 4z - 5 = 0$.

Question

Under what conditions on α do the spheres $x^2 + y^2 + z^2 + \alpha x - y = 0$ and $x^2 + y^2 + z^2 + x + 2z + 1 = 0$
intersect each other at an angle of 45 degree?

Solution

$S_1: x^2 + y^2 + z^2 + \alpha x - y = 0$
or
$$S_1: \left(x + \frac{\alpha}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \frac{\alpha^2 + 1}{4}$$
Its centre is $\left(-\frac{\alpha}{2}, \frac{1}{2}, 0\right)$ and radius $= \frac{\sqrt{\alpha^2 + 1}}{2}$.

$S_2: x^2 + y^2 + z^2 + x + 2z + 1 = 0$
or
$$S_2: \left(x + \frac{1}{2}\right)^2 + y^2 + (z + 1)^2 = \frac{1}{4}$$
Its centre is $\left(-\frac{1}{2}, 0, -1\right)$ and radius $= \frac{1}{2}$.
The intersection plane is obtained as $S_2 - S_1 = 0$
$x^2 + y^2 + z^2 + x + 2z + 1 - (x^2 + y^2 + z^2 + \alpha x - y) = 0$
or
$(1 - \alpha)x + y + 2z + 1 = 0$
The sphere's center distance
$$d = \sqrt{\left(-\frac{\alpha}{2} - \left(-\frac{1}{2}\right)\right)^2 + \left(\frac{1}{2} - 0\right)^2 + (0 - (-1))^2}$$
Consider the intersection point as the first vertice and the sphere centers as the remaining two triangle vertices. The Law of Cosines
$d^2 = {r_1}^2 + {r_2}^2 - 2r_1r_2\cos\theta$
If $\theta = 45°$
$$\frac{(1 - \alpha)^2}{4} + \frac{1}{4} + 1 = \frac{\alpha^2 + 1}{4} + \frac{1}{4} - 2\left(\frac{\sqrt{\alpha^2 + 1}}{2}\right)\left(\frac{1}{2}\right)\cos 45°$$
$\sqrt{2}\sqrt{\alpha^2 + 1} = \alpha^2 + 1 - 1 + 2\alpha - \alpha^2 - 4$
$\sqrt{2}\sqrt{\alpha^2 + 1} = 2\alpha - 4, 2\alpha - 4 > 0 => \alpha > 2$
$2(\alpha^2 + 1) = 4(\alpha - 2)^2$
$2\alpha^2 - 8\alpha + 8 - \alpha^2 - 1 = 0$
$\alpha^2 - 8\alpha + 7 = 0$

$$\alpha = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(7)}}{2(1)} = 4 \pm 3$$

$\alpha_1 = 4 - 3 = 1, \;\; \alpha_2 = 4 + 3 = 7$

Since $\alpha > 2$, we take $\alpha = 7$.

Answer: $\alpha = 7$.