Answer on Question #82444 – Math – Calculus

Question

Evaluate $\Gamma(-3/2)$

Solution

For $x > 0$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

Then

$$\Gamma(x+1) = \int_0^\infty t^{x+1-1} e^{-t}\, dt = \int_0^\infty t^{x} e^{-t}\, dt$$

Integrate by parts

$$\Gamma(x+1) = \int_0^\infty t^{x+1-1} e^{-t}\, dt = \int_0^\infty t^{x} e^{-t}\, dt = \left[-t^x e^{-t}\right]_0^\infty + x\int_0^\infty t^{x-1} e^{-t}\, dt =$$

$$= 0 + x\int_0^\infty t^{x-1} e^{-t}\, dt = x\Gamma(x)$$

Functional equation

$$\Gamma(x+1) = x\Gamma(x), x > 0$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1} e^{-t}\, dt$$

The change of variable $t = u^2$ gives

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1} e^{-t}\, dt = 2\int_0^\infty e^{-u^2}\, du = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$$

We can extend $\Gamma(x)$ to non-integer negative values by defining

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, x < 0, x \notin Z$$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) = -\frac{2}{3}\left(\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}\right) = \frac{4}{3}\Gamma\left(\frac{1}{2}\right) = \frac{4\sqrt{\pi}}{3}$$

Answer: $\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3}$.