Answer on Question #82283 – Math – Differential Equations

Question

(x^3 - 1)y''(x) + (x^2)y'(x) + xy(x) = 0, y(0)=2, y'(0)=1

Solution

This equation of the form:

$$x^2(ax^n - 1)y''(x) + x(apx^n + q)y'(x) + (arx^n + s)y(x) = 0$$

where n=3, a=1, p=1, q=0, r=1, s=0. Find the roots of the quadratic equations:

$$\begin{cases} A^2 - (q+1)A - s = 0 \\ B^2 - (p-1)B + r = 0 \end{cases} \rightarrow \begin{cases} A^2 - A = 0 \\ B^2 + 1 = 0 \end{cases} \rightarrow A_1 = 0, A_2 = 1, B_1 = -i, B_2 = i$$

And define parameters c,α,β and γ:

$$c = A_1 = 0$$

$$\alpha = (A_1 + B_1)n^{-1} = \frac{-i}{3}$$

$$\beta = (A_1 + B_2)n^{-1} = \frac{i}{3}$$

$$\gamma = 1 + (A_1 - A_2)n^{-1} = \frac{2}{3}$$

Then the solution of the original equation has the form $y(x) = x^c u(ax^n) = u(x^3)$ where u=u(z) is the general solution of the hypergeometric equation:

$$z(z-1)u''(z) + \big((\alpha + \beta + 1)z - \gamma\big)u'(z) + \alpha\beta u(z) = 0$$

$$z(z-1)u''(z) + \left(z - \frac{2}{3}\right)u'(z) + u(z)/9 = 0$$

γ=2/3 is not an integer, the general solution of the hypergeometric equation has the form:

$$u(z) = C_1 F(\alpha, \beta, \gamma; z) + C_2 z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z)$$

$$y(x) = u(x^3) = C_1 \ {}_2F_1\left(\frac{-i}{3}, \frac{i}{3}, \frac{2}{3}; x^3\right) + C_2 x \ {}_2F_1\left(\frac{-i}{3} + \frac{1}{3}, \frac{i}{3} + \frac{1}{3}, \frac{4}{3}; x^3\right)$$

where ${}_2F_1$ is the hypergeometric function

$${}_2F_1(l, m, k; t) = \sum_{n=0}^{\infty} \frac{(l)_n (m)_n}{(k)_n} \frac{t^n}{n!}$$

$$\frac{d}{dt}\left({}_2F_1(l, m, k; t)\right) = \frac{lm}{k} \ {}_2F_1(l+1, m+1, k+1; t)$$

For y(0)=2, y'(0)=1:

$$y(0) = 2 = C_1 \ {}_2F_1\left(\frac{-i}{3}, \frac{i}{3}, \frac{2}{3}; 0\right) + C_2 0 \ {}_2F_1(\frac{-i}{3} + \frac{1}{3}, \frac{i}{3} + \frac{1}{3}, \frac{4}{3}; 0) \to C_1 = 2$$

$$y'(x) = C_1 \frac{d}{dx}(\ {}_2F_1\left(\frac{-i}{3}, \frac{i}{3}, \frac{2}{3}; x^3\right)) + C_2 \frac{dx}{dx} \ {}_2F_1(\frac{-i}{3} + \frac{1}{3}, \frac{i}{3} + \frac{1}{3}, \frac{4}{3}; x^3)$$
$$+ C_2 x \frac{d}{dx}(\ {}_2F_1(\frac{-i}{3} + \frac{1}{3}, \frac{i}{3} + \frac{1}{3}, \frac{4}{3}; x^3))$$
$$= 2\frac{1}{9}\frac{3}{2} \ {}_2F_1\left(\frac{-i}{3} + 1, \frac{i}{3} + 1, \frac{5}{3}; x^3\right) 3x^2 + C_2 \ {}_2F_1(\frac{-i}{3} + \frac{1}{3}, \frac{i}{3} + \frac{1}{3}, \frac{4}{3}; x^3)$$
$$+ C_2 x \frac{2}{9}\frac{3}{4}(\ {}_2F_1(\frac{-i}{3} + \frac{4}{3}, \frac{i}{3} + \frac{4}{3}, \frac{7}{3}; x^3))3x^2 \ \to y'(0) = 1 \to C_2 = 1$$

$$y(x) = 2 \ {}_2F_1\left(\frac{-i}{3}, \frac{i}{3}, \frac{2}{3}; x^3\right) + x \ {}_2F_1(\frac{-i}{3} + \frac{1}{3}, \frac{i}{3} + \frac{1}{3}, \frac{4}{3}; x^3)$$

Answer: $y(x) = 2 \ {}_2F_1\left(\frac{-i}{3}, \frac{i}{3}, \frac{2}{3}; x^3\right) + x \ {}_2F_1(\frac{-i}{3} + \frac{1}{3}, \frac{i}{3} + \frac{1}{3}, \frac{4}{3}; x^3)$.