Answer on Question #82172 – Math – Calculus

Let p and q be real numbers and let f be the function defined by:

$$f(x) = \begin{cases} 1 + 2p(x-1) + (x-1)^2, & x \le 1 \\ qx + p, & x > 1 \end{cases}$$

Question

a. Find the values of q in terms of p for which f is continuous at $x = 1$.

Solution

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1 + 2p(x-1) + (x-1)^2) = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (qx + p) = q + p$

If $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$, then $\lim_{x\to 1} f(x)$ exists and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x)$$

Hence,

$$q + p = 1$$

$$\lim_{x\to 1} f(x) = 1$$

$f(1) = 1 + 2p(1-1) + (1-1)^2 = 1 = \lim_{x\to 1} f(x)$

The function f is continuous at $x = 1$ if $q = 1 - p$

Answer: the function f is continuous at $x = 1$ if $q = 1 - p$.

Question

b. Find the values of q in terms of p for which f is differentiable at $x = 1$.

Solution

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} =$$

$$= \lim_{h\to 0^-} \frac{1 + 2p(1+h-1) + (1+h-1)^2 - (1 + 2p(1-1) + (1-1)^2)}{h} =$$

$$= \lim_{h\to 0^-} \frac{1 + 2ph + h^2 - 1}{h} = \lim_{h\to 0^-} (2p + h) = 2p$$

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{q(1+h) + p - 1}{h} =$$

$$= \lim_{h\to 0^-} \frac{q + p - 1}{h} + q$$

If f is differentiable at $x = 1$, then

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$2p = \lim_{h\to 0^-} \frac{q+p-1}{h} + 1$$

$$\begin{cases} q+p-1=0 \\ q=2p \end{cases} => \begin{cases} 2p+p=1 \\ q=2p \end{cases} => \begin{cases} p=\frac{1}{3} \\ q=\frac{2}{3} \end{cases}$$

Answer: $q = \frac{2}{3}, p = \frac{1}{3}$.

Question

c. If p and q have the values determined in part (b) is f'' a continuous function?

Solution

$$\begin{cases} p=\frac{1}{3} \\ q=\frac{2}{3} \end{cases}$$

$$f(x) = \begin{cases} 1+\frac{2}{3}(x-1)+(x-1)^2, & x\le 1 \\ \frac{2}{3}x+\frac{1}{3}, & x>1 \end{cases}$$

$$f'(x) = \begin{cases} \frac{2}{3}+2(x-1), & x\le 1 \\ \frac{2}{3}, & x>1 \end{cases}$$

$$f'(x) = \begin{cases} 2x-\frac{4}{3}, & x\le 1 \\ \frac{2}{3}, & x>1 \end{cases}$$

$f''(x) = 2, x < 1$
$f''(x) = 0, x > 1$

$$\lim_{h\to 0^-} \frac{f'(1+h)-f'(1)}{h} = \lim_{h\to 0^-} \frac{2(1+h)-\frac{4}{3}-\left(2(1)-\frac{4}{3}\right)}{h} = 2$$

$$\lim_{h\to 0^+} \frac{f'(1+h)-f'(1)}{h} = \lim_{h\to 0^+} \frac{\frac{2}{3}-\left(2(1)-\frac{4}{3}\right)}{h} = 0$$

$Since \lim_{h\to 0^-} \frac{f'(1+h)-f'(1)}{h} = 2 \neq 0 = \lim_{h\to 0^+} \frac{f'(1+h)-f'(1)}{h}$, then

$$\lim_{h\to 0}\frac{f'(1+h)-f'(1)}{h} \text{ does not exist}$$

The function $f'(x)$ is not differentiable at $x=1$.

$$f''(x)=\begin{cases}2, & x<1\\ 0, & x>1\end{cases}$$

Therefore, $f''(x)$ is not continuous at $x=1$.

Answer: $f''(x)$ is not continuous at $x=1$.