Answer on Question #81421 – Math – Linear Algebra

Question

Consider the basis $e_1 = (-2, 4, -1), e_2 = (-1, 3, -1)$ and $e_3 = (1, -2, 1)$ of $\mathbb{R}^3$ over $\mathbb{R}$. Find the dual basis of $\{e_1, e_2, e_3\}$.

Solution

Let $\{e^1, e^2, e^3\}$ is the dual basis of $\{e_1, e_2, e_3\}$.Then $\langle e^i, e_j \rangle = \delta^i_j$ or

$$(e^1 \quad e^2 \quad e^3)\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = I_3$$

$$If \;\; A = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -2 & 4 & -1 \\ -1 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} \; then \; A^{-1} = (e^1 \quad e^2 \quad e^3)$$

$$A^{-1}A = I_3$$

Augment the matrix A with identity matrix

$$\begin{pmatrix} -2 & 4 & -1 & 1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 4 & -1 & 1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1/(-2)} \begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3-R_1} \begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix} \xrightarrow{R_1+(2)R_2} \begin{pmatrix} 1 & 0 & -1/2 & -3/2 & 2 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1/2 & -1/2 & 0 & 0 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix} \xrightarrow{R_1+R_3} \begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+R_3} \begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1/2 & 1/2 & 0 & 1 \end{pmatrix} \xrightarrow{(2)R_3} \begin{pmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{pmatrix}$$

As can be seen, we have obtained the identity matrix to the left. So

$$A^{-1} = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} = (e^1 \quad e^2 \quad e^3)$$

Therefore, the vectors

$$e^1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

$$e^2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix},$$

$$e^3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

form the dual basis.

Answer:

$$e^1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \; e^2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \; e^3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$