Answer on Question #81384 – Math – Linear Algebra

which of the following statements are true and which are false? justify your answer with a short proof or a counterexample.

Question

(i) $\mathbb{R}^2$ has infinitely many nonzero, proper vector subspaces

Solution

Let $\{e_1, e_2\}$ be a basis in $\mathbb{R}^2$. Then

$W_n = \{\alpha(e_1 + ne_2), \alpha \in \mathbb{R}\}, n \in \mathbb{N}$

are all different proper vector subspaces. Indeed, suppose $W_n = W_m$. Then

$e_1 + ne_2 = \alpha(e_1 + me_2)$

for some α.

Then

$(\alpha - 1)e_1 + (m\alpha - n)e_2 = 0$

from which $\alpha = 1, \alpha = n/m$, i.e. $n = m$.

This proves that W_n are all different. And there are infinitely many of them.

Conclusion: $\mathbb{R}^2$ has infinitely many nonzero, proper vector subspaces – true.

Question

ii) if $T: V \to W$ is one-one linear transformation between two finite dimensional vector spaces V and W then T is invertible

Solution

Let T be one-to one. For any $y \in W$ define Sy as the (unique) solution to $Tx = y$.

Then obviously

$S(Tx) = x$.

It is sufficient to prove that S is linear.

$$S(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 S y_1 + \alpha_2 S y_2$$

Let $Sy_1 = x_1, Sy_2 = x_2$. Then

$T(\alpha_1 x_1 + \alpha_2 x_2) = T(\alpha_1 S y_1 + \alpha_2 S y_2) = \alpha_1 T(S y_1) + \alpha_2 T(S y_2) = \alpha_1 y_1 + \alpha_2 y_2$,

from which

$$\alpha_1 x_1 + \alpha_2 x_2 = S(\alpha_1 y_1 + \alpha_2 y_2)$$

which proves linearity.

Conclusion: if $T: V \to W$ is one-one linear transformation between two finite dimensional vector spaces V and W then T is invertible - true.

Question

iii) if A^k = 0 for a square matrix A, then all the eigen values of a are nonzero

Solution

Consider $A = 0$. It has all zero eigenvalues.

Conclusion: if A^k = 0 for a square matrix A, then all the eigen values of a are nonzero – false.

Question

iv) every unitary operator is invertible

Solution

By definition an unitary operator U is such that

$UU^* = U^*U$,

thus U^* is inverse of U.

Conclusion: every unitary operator is invertible – true.

Question

v) every system of homogeneous linear equations has a nonzero solution

Solution

Consider a system:

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases}$$

It has only zero solution.

Conclusion: every system of homogeneous linear equations has a nonzero solution – false.

Answer:

(i) $\mathbb{R}^2$ has infinitely many non zero, proper vector subspaces – true;

ii) if $T: V \to W$ is one-one linear transformation between two finite dimensional vector spaces

V and W then T is invertible – true;

iii) if A^k = 0 for a square matrix A, then all the eigen values of a are nonzero – false;

iv) every unitary operator is invertible – true;

v) every system of homogeneous linear equations has a nonzero solution – false.