Answer on Question #81171 – Math – Abstract Algebra

Question

Prove that R^(n)/R^(m) ~ R^(n-m) as groups, where n, m∈ N, n ≥ m.

Solution

Consider φ: $R^n \to R^{n-m}$:

$$\varphi\big((v_1, v_2, \dots, v_n)\big) = (v_1, v_2, \dots, v_{n-m}) \text{ for } (v_1, v_2, \dots, v_n) \in R^n.$$

We check that φ is a homomorphism:

$$\begin{aligned}\varphi\big((v_1, v_2, \dots, v_n) + (u_1, u_2, \dots, u_n)\big) &= \varphi\big((v_1 + u_1, v_2 + u_2, \dots, v_n + u_n)\big)\\ &= (v_1 + u_1, v_2 + u_2, \dots, v_{n-m} + u_{n-m})\\ &= (v_1, v_2, \dots, v_{n-m}) + (u_1, u_2, \dots, u_{n-m})\\ &= \varphi\big((v_1, v_2, \dots, v_n)\big) + \varphi\big((u_1, u_2, \dots, u_n)\big)\end{aligned}$$

for $(v_1, v_2, \dots, v_n), (u_1, u_2, \dots, u_n) \in R^n$

Remark that $\ker(\varphi) = \{(0, \dots, 0, v_{n-m}, \dots, v_n)\}$ for all $(v_{n-m}, \dots, v_n) \in R^m$ => $\ker(\varphi) \triangleq R^m$

φ is surjective: if $(w_1, w_2, \dots, w_{n-m}) \in R^{n-m}$ then $\varphi\big((w_1, w_2, \dots, w_{n-m}, 0, \dots, 0)\big)$ $= (w_1, w_2, \dots, w_{n-m})$ =>

=> $\operatorname{im}(\varphi) = R^{n-m}$

By the first isomorphism theorem, $R^n / \ker(\varphi) \triangleq \operatorname{im}(\varphi) \iff R^n / R^m \triangleq R^{n-m}$.