Answer on Question #81144 – Math – Linear Algebra

Question

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1 - x_3, x_2 - x_3, x_1)$.
Is T invertible? If yes, find a rule for T^{-1} like the one which defines T.

Solution

If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x_1, x_2, x_3) = (x_1 - x_3, x_2 - x_3, x_1)$, we can define

$$T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$Det(T) = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} - 0 - 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 + 0 - (0 - 1) = 1 \neq 0$$

Then T is invertible.
Augment the matrix T with identity matrix

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + R_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + R_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

As can be seen, we have obtained the identity matrix to the left. So, we are done.

$$T^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$T^{-1}: \mathbb{R}^3 \to \mathbb{R}^3$ can be defined by

$$T^{-1}(x_1, x_2, x_3) = (x_3, -x_1 + x_2 + x_3, -x_1 + x_3).$$