

## Answer on Question #80846 – Math – Linear Algebra

### Question

Consider the linear operator  $T: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ , defined by  $T(z_1, z_2, z_3) = (z_1 - iz_2, iz_1 + 2z_2 + iz_3, -iz_2 + z_3)$ .

- i) Compute  $T^*$  and check whether  $T$  is selfadjoint.
- ii) Check whether  $T$  is unitary.

### Solution

i) Given the rules

$$z_1 \rightarrow z_1 - iz_2 = w_1$$

$$z_2 \rightarrow iz_1 + 2z_2 + iz_3 = w_2$$

$$z_3 \rightarrow -iz_2 + z_3 = w_3$$

$$w = Tz$$

For linear operator  $T$  the matrix representation is

$$T = \begin{bmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{bmatrix}$$

We recall, that an operator  $T^*$  is called adjoint for the linear operator  $T$  if for all  $x, y \in \mathbb{C}^3$   $(Tx, y) = (x, T^*y)$ . The matrix representation for  $T^*$  can be found as

$$T^* = (\overline{T})^T = \overline{(T^T)}$$

where  $A^T$  denotes the transpose and  $\overline{A}$  denotes the matrix with complex conjugated entries.

In our case

$$T^* = \begin{bmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{bmatrix} = T$$

The adjoint operator  $T^*(z_1, z_2, z_3) = (z_1 - iz_2, iz_1 + 2z_2 + iz_3, -iz_2 + z_3)$ . Therefore,  $T$  is selfadjoint.

ii) A unitary operator is a bounded linear operator on a Hilbert space that satisfies  $U^*U = UU^* = I$  where  $U^*$  is the adjoint of  $U$ .

$$\begin{aligned} T \cdot T^* &= \begin{bmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{bmatrix} = \\ &= \begin{bmatrix} 1(1) - i(i) + 0(0) & 1(-i) - i(2) + 0(-i) & 1(0) - i(i) + 0(1) \\ i(1) + 2(i) + i(0) & i(-i) + 2(2) + i(-i) & i(0) + 2(i) + i(1) \\ 0(1) - i(i) + 1(0) & 0(-i) - i(2) + 1(-i) & 0(0) - i(i) + 1(1) \end{bmatrix} = \\ &= \begin{bmatrix} 2 & -3i & 1 \\ 3i & 6 & 3i \\ 1 & -3i & 2 \end{bmatrix} \neq I_3 \end{aligned}$$

Therefore,  $T$  is not unitary.

**Answer:**

**i)**  $T$  is selfadjoint.

**ii)**  $T$  is not unitary.