Question #8076 Interpret and prove: 1)$o + o = o$, 2)$o + O = O$, 3)$O + O = O$, 4)$o(O) = o$, 5)$o \cdot O = o$. **Solution.** Through the solution for simplicity of notation I will use $f_i = f_i(x)$, $g = g(x)$.

1) If $f_1 = o(g), f_2 = o(g), x \to x_0$, then $f_1 + f_2 = o(g), x \to x_0$. The condition implies that $f_i(x)/g(x) \to 0, x \to x_0$, $i = 1, 2$, hence $\frac{f_1+f_2}{g} \to 0, t \to x_0$, thus $f_1 + f_2 = o(g), x \to x_0$.

2)if $f_1 = o(g), f_2 = O(g), x \to x_0$ then $f_1 + f_2 = O(g), x \to x_0$. It is obvious that if $f_1 = o(g), x \to x_0$ entails $f_1 = O(g)$, hence the statement of 2) will follow from 3).

3) $f_i = O(g), x \to x_0$, $i = 1, 2$ then $f_1 + f_2 = O(g), x \to x_0$. The condition implies exist δ_i such that for all $x \in B_{\delta_i}(x_0)$ $|f_i(x)| < K_i|g(x)|$, $i = 1, 2$. Take $\delta := \min\{\delta_1, \delta_2\}$, then those two inequalities hold.Hence for all $x \in B_\delta(x)$ $|f_1 + f_2| \leq (K_1 + K_2)|g(x)|$, thus $f_1 + f_2 = O(g)$.

4) If $f_1 = O(f_2), g = o(f_1), x \to x_0$, then $g = o(f_2), x \to x_0$. For all ϵ exists such δ_1 that $|g(x)| < \epsilon|f_1(x)|$, $x \in B_{\delta_1}(x_0)$. Moreover, exists δ_2 and $K > 0$ such that for all $x \in B_{\delta_2}(x_0)$ $|f_1| \leq K|f_2|$, hence for all $x \in B_{\min\{\delta_1,\delta_2\}}(x_0)$ $|g| < K\epsilon|f_2|$ and we are done

5) Can be proven in the same manner as 4).