Answer on Question #80396 – Math – Calculus

Question

Find the surface area of the solid formed by the rotation of an arc of the cycloid $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$ about the axis of x.

Solution

The parametric equation of cycloid is
$x(\theta) = a(\theta + \sin\theta), y(\theta) = a(1 + \cos\theta), \theta \in [-\pi, \pi]$
Suppose $a > 0$. Then $y \geq 0$
Its surface of revolution about the axis of x is given by

$$S: 2\pi \int_{\theta_1}^{\theta_2} y(\theta)\sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta$$

$x'(\theta) = \big(a(\theta + \sin\theta)\big)' = a(1 + \cos\theta)$
$y'(\theta) = \big(a(1 + \cos\theta)\big)' = -a\sin\theta$
$(x'(\theta))^2 + (y'(\theta))^2 = (a(1 + \cos\theta))^2 + (-a\sin\theta)^2 =$
$= a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta) = 2a^2(1 + \cos\theta) = 2a^2\left(2\cos^2\left(\frac{\theta}{2}\right)\right) =$
$= 4a^2\cos^2\left(\frac{\theta}{2}\right)$

$S = 2\pi \int_{\theta_1}^{\theta_2} y(\theta)\sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta =$

$= 2\pi \int_{-\pi}^{\pi} a(1 + \cos\theta)\sqrt{4a^2\cos^2\left(\frac{\theta}{2}\right)}\, d\theta$

$= 2\pi \int_{-\pi}^{\pi} a\left(2\cos^2\left(\frac{\theta}{2}\right)\right)\left(2a\cos\left(\frac{\theta}{2}\right)\right) d\theta$

$= 8\pi a^2 \int_{-\pi}^{\pi} \cos^3\left(\frac{\theta}{2}\right) d\theta$

$\int \cos^3\left(\frac{\theta}{2}\right) d\theta$

Substitution

$u = \sin\left(\frac{\theta}{2}\right), du = \frac{1}{2}\cos\left(\frac{\theta}{2}\right) d\theta$

$\cos^2\left(\frac{\theta}{2}\right) = 1 - \sin^2\left(\frac{\theta}{2}\right) = 1 - u^2$

$\int \cos^3\left(\frac{\theta}{2}\right) d\theta = 2\int (1 - u^2)\, du = 2\left(u - \frac{u^3}{3}\right) + C =$

$= 2\left(\sin\left(\frac{\theta}{2}\right) - \frac{\sin^3\left(\frac{\theta}{2}\right)}{3}\right) + C$

$$S = 2\pi \int_{\theta_1}^{\theta_2} y(\theta)\sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, d\theta =$$

$$= 8\pi a^2 \int_{-\pi}^{\pi} \cos^3\left(\frac{\theta}{2}\right) d\theta =$$

$$= 16\pi a^2 \left[\sin\left(\frac{\theta}{2}\right) - \frac{\sin^3\left(\frac{\theta}{2}\right)}{3}\right]_{-\pi}^{\pi} =$$

$$= 16\pi a^2 \left(\sin\left(\frac{\pi}{2}\right) - \frac{\sin^3\left(\frac{\pi}{2}\right)}{3} - \left(\sin\left(\frac{-\pi}{2}\right) - \frac{\sin^3\left(\frac{-\pi}{2}\right)}{3}\right)\right) =$$

$$= 16\pi a^2 \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \frac{64\pi a^2}{3}\ (units^2)$$

$$S = \frac{64\pi a^2}{3}\ square\ units$$