Answer on Question #80359 – Math – Linear Algebra

Question

Reduce the conic $x^2 + 6xy + y^2 - 8 = 0$.

Solution

The General Equation for a Conic Section

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

In the given case we have $x^2 + 6xy + y^2 = 0$

$$A = 1, B = 6, C = 1, D = 0, E = 0, F = -8$$

$$B^2 - 4AC = (36)^2 - 4(1)(1) = 32 > 0$$

Then we have hyperbola or 2 intersecting lines.

A conic equation of the type of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is rotated by an angle θ, to form a new Cartesian plane with coordinates (x', y'), if θ is appropriately chosen, we can have a new equation without term xy i.e. of standard form.

The relation between coordinates (x, y) and (x', y') can be expressed as

$$x = x' \cos\theta - y' \sin\theta, \qquad y = x' \sin\theta + y' \cos\theta$$

or

$$x' = x \cos\theta + y \sin\theta, \qquad y' = -x \sin\theta + y \cos\theta$$

For this we need to have θ given by

$$\cot 2\theta = \frac{A - C}{B}$$

$A = 1, B = 6, C = 1$

$$\cot 2\theta = \frac{1 - 1}{6} = 0 => \theta = \frac{\pi}{4}$$

Then

$$x = x' \cos\frac{\pi}{4} - y' \sin\frac{\pi}{4}, \qquad y = x' \sin\frac{\pi}{4} + y' \cos\frac{\pi}{4}$$

$$x = x' \frac{1}{\sqrt{2}} - y' \frac{1}{\sqrt{2}}, \qquad y = x' \frac{1}{\sqrt{2}} + y' \frac{1}{\sqrt{2}}$$

$$x^2 + 6xy + y^2 = 8$$

$$\left(x' \frac{1}{\sqrt{2}} - y' \frac{1}{\sqrt{2}}\right)^2 + 6\left(x' \frac{1}{\sqrt{2}} - y' \frac{1}{\sqrt{2}}\right)\left(x' \frac{1}{\sqrt{2}} + y' \frac{1}{\sqrt{2}}\right) + \left(x' \frac{1}{\sqrt{2}} + y' \frac{1}{\sqrt{2}}\right)^2 = 8$$

$$\frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2 + 6\left(\frac{1}{2}\right)x'^2 - 6\left(\frac{1}{2}\right)y'^2 + \frac{1}{2}x'^2 + x'y' + \frac{1}{2}y'^2 = 8$$

$$4x'^2 - 2y'^2 = 8$$

$$\frac{x'^2}{2} - \frac{y'^2}{4} = 1$$

This is a canonical equation of hyperbola.

Or

$x^2 + 6xy + y^2 = 8$

$x^2 + 2x(3y) + (3y)^2 - (3y)^2 + y^2 = 8$

$(x + 3y)^2 - 8y^2 = 8$

Substituting $x' = x + 3y, y' = y$ we obtain

${x'}^2 - 8{y'}^2 = 8$

$\frac{{x'}^2}{8} - \frac{{y'}^2}{1} = 1$

This is a canonical equation of hyperbola.