Answer on Question #80341 – Math – Calculus

Question

Prove that for every x greater than 0, x /1 +x2 less than tan inverse x less than x

Solution

We can prove this inequality by using the Mean Value Theorem:

If $f(x)$ *is continuous on the closed interval* $[a, b]$ *and differentiable on the open interval* (a, b)*, there exists a number* $c \in (a, b)$ *such that* $f'(c) = \frac{f(b)-f(a)}{b-a}$.

We want to prove the inequality $\frac{x}{1+x^2} < \arctan x < x$.

Let $x > 0$. Applying the Mean Value Theorem to the function $f(x) = \arctan x$ on the interval $[0, x]$ gives a number c such that

$$f'(c) = (\arctan x)'|_{x=c} = \left.\frac{1}{1+x^2}\right|_{x=c} = \frac{1}{1+c^2}.$$

So we have

$$\frac{1}{1+c^2} = \frac{\arctan x - \arctan 0}{x - 0};$$

$$\frac{1}{1+c^2} = \frac{\arctan x}{x};$$

$$\arctan x = \frac{x}{1+c^2}.$$

We remember that $c \in (0, x)$, this gives us $0 < c < x$, so we have the following inequality

$$\frac{x}{1+x^2} < \frac{x}{1+c^2} < x;$$

and finally

$$\frac{x}{1+x^2} < \arctan x < x.$$

Proof is complete.