

ANSWER on Question #80315 – Math – Calculus

QUESTION

Is the given statement is true or false:

Every integrable function is differentiable

SOLUTION

Hint: under the expression "integrable function" we mean "Riemann integrable".

Integrable over Riemann on an interval $[a, b]$ means that

$$\exists A < \infty : \int_a^b f(x) dx = A$$

(More information: https://en.wikipedia.org/wiki/Riemann_integral)

Recall the definition of the differentiability of a function.

The function $f(x)$ is differentiable on an interval $[a, b]$ if there exists a limit

$$\forall x \in [a, b] : a = f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

(More information : https://en.wikipedia.org/wiki/Differentiable_function)

Here is a counterexample to this statement, that is, give an example of the functions are integrable but differentiable.

For example,

$$f(x) = |x| \text{ at } x \in [-1, 1]$$

Then,

$$\begin{aligned} \int_{-1}^1 |x| dx &= \int_{-1}^0 |x| dx + \int_0^1 |x| dx = \int_{-1}^0 (-x) dx + \int_0^1 x dx = \left(-\frac{x^2}{2}\right) \Big|_{-1}^0 + \left(\frac{x^2}{2}\right) \Big|_0^1 = \\ &= -\frac{0^2}{2} - \left(-\frac{(-1)^2}{2}\right) + \frac{1^2}{2} - \frac{0^2}{2} = -\frac{0}{2} + \frac{1}{2} + \frac{1}{2} - \frac{0}{2} = 1 \end{aligned}$$

Conclusion,

$$\int_{-1}^1 |x| dx = 1$$

Now, let us investigate the differentiability of the function $f(x) = |x|$ at the point $x = 0$.

$$f'(0) = \lim_{x \rightarrow 0} \frac{|x| - |0|}{x - 0} = \lim_{x \rightarrow 0} \frac{|x| - 0}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x}$$

Since the absolute value can be opened depending on the sign of the inside expression, we consider two cases.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

1 case: $x > 0$

$$f'(0) = \lim_{x \rightarrow 0} \frac{|x|}{x} = \lim_{x \rightarrow 0} \left(\frac{x}{x}\right) = \lim_{x \rightarrow 0} (1) = 1 \rightarrow \boxed{f'(0) = 1}$$

2 case: $x < 0$

$$f'(0) = \lim_{x \rightarrow 0} \frac{|x|}{x} = \lim_{x \rightarrow 0} \left(\frac{-x}{x}\right) = \lim_{x \rightarrow 0} (-1) = -1 \rightarrow \boxed{f'(0) = -1}$$

Conclusion,

$$\boxed{f(x) = |x| \text{ is not differentiable at the point } x = 0}$$

General conclusion,

$$\boxed{\text{Every integrable function is differentiable} - \text{FALSE}}$$

ANSWER

Every integrable function is differentiable – FALSE