ANSWER on Question #80313 – Math – Algebra

QUESTION

Find the domain and range of

$$f(x) = \sqrt{\frac{(1-x^2)}{(x-2)}}$$

SOLUTION

By the definition, the domain of a function is the set of “input” or argument values for which the function is defined.

(More information: https://en.wikipedia.org/wiki/Domain_of_a_function)

In our case,

$$f(x) = \sqrt{\frac{(1-x^2)}{(x-2)}} - square\ root$$

The square root property that we need to define a domain function:

$$\boxed{\sqrt{a}\ is\ defined\ for\ any\ a \geq 0}$$

(More information: https://en.wikipedia.org/wiki/Square_root)

Then,

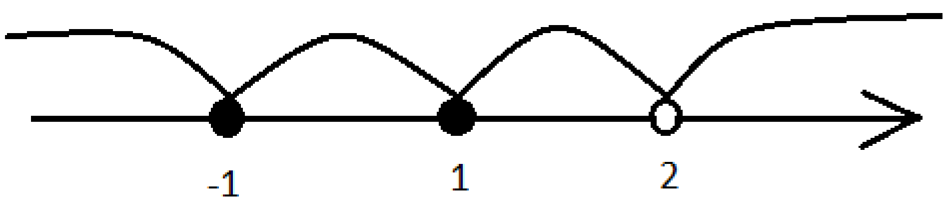
$$\frac{(1-x^2)}{(x-2)} \geq 0 \rightarrow \frac{(-1)\cdot(x^2-1)}{(x-2)} \geq 0 \Bigg| \times (-1) \rightarrow \boxed{\frac{(x-1)\cdot(x+1)}{(x-2)} \leq 0}$$

It remains to solve the highlighted inequality.

1 STEP: Let us find the roots of the function

$$g(x) = \frac{(x-1)\cdot(x+1)}{(x-2)} \rightarrow \frac{(x-1)\cdot(x+1)}{(x-2)} = 0 \rightarrow \left[\begin{array}{l} x-1=0 \\ x+1=0 \\ x-2 \neq 0 \end{array}\right. \rightarrow \boxed{\left[\begin{array}{c} x=1 \\ x=-1 \\ x \neq 2 \end{array}\right.}$$

2 STEP: We note the roots found on the real line



As we can see, the roots found split the whole number line into four intervals:

$$1 : x \in (-\infty, -1]$$

$$2 : x \in [-1,1]$$

$$3 : x \in [1,2)$$

$$4 : x \in (2, +\infty)$$

(More information: https://en.wikipedia.org/wiki/Interval_(mathematics))

3 STEP: We define the sign of the function $g(x)$ on each of the intervals by the trial point method.

$$g(x) = \frac{(x-1)\cdot(x+1)}{(x-2)} \rightarrow$$

$$1 : x \in (-\infty, -1] \rightarrow g(x=-2) = \frac{(-2-1)\cdot(-2+1)}{-2-2} = \frac{(-3)\cdot(-1)}{-4} = \frac{3}{-4} = -0.75 < 0 \rightarrow$$

$$\boxed{g(x) < 0, \forall x \in (-\infty, -1]}$$

$$2 : x \in [-1,1] \rightarrow g(x=0) = \frac{(0-1)\cdot(0+1)}{0-2} = \frac{(-1)\cdot(1)}{-2} = \frac{-1}{-4} = 0.5 > 0 \rightarrow$$

$$\boxed{g(x) > 0, \forall x \in [-1,1]}$$

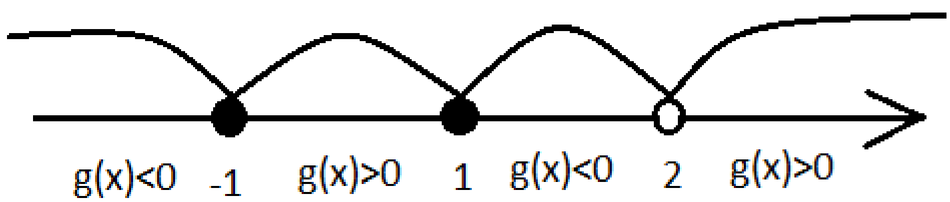
$$1 : x \in [1,2) \rightarrow g(x=1.5) = \frac{(1.5-1)\cdot(1.5+1)}{1.5-2} = \frac{(0.5)\cdot(2.5)}{-0.5} = -2.5 < 0 \rightarrow$$

$$\boxed{g(x) < 0, \forall x \in [1,2)}$$

$$1 : x \in (2, +\infty) \rightarrow g(x=3) = \frac{(3-1)\cdot(3+1)}{3-2} = \frac{(2)\cdot(4)}{1} = 6 > 0 \rightarrow$$

$$\boxed{g(x) > 0, \forall x \in (2, +\infty)}$$

Visually it looks like this



Conclusion,

Since we needed to solve the inequality $g(x) \leq 0$, then from all the intervals we have to choose those where condition $g(x) < 0$ holds. Then,

$$\boxed{f(x) = \sqrt{\frac{(1-x^2)}{(x-2)}} \rightarrow D(f) : x \in (-\infty, -1]\text{U}[1,2)\ is\ the\ domain\ of\ a\ function}$$

By the definition, the range of a function $E(f)$ is the set of such numbers y such that for any number $x \in D(f)$ the equality

$$y = f(x)$$

or

$$E(f) = \{y | \forall x \in D(f) : y = f(x)\}$$

(More information: https://en.wikipedia.org/wiki/Range_(mathematics))

In our case,

$$\begin{cases} f(x) = \sqrt{\dfrac{(1-x^2)}{(x-2)}} \\ D(f) : x \in (-\infty, -1]\text{U}[1,2) \end{cases}$$

At each of the two intervals of the domain of definition, this function is continuous.

Then, in order to find the range function, it is sufficient to find the limits of the function at the ends by an interval.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \sqrt{\frac{(1-x^2)}{(x-2)}} = +\infty$$

$$\lim_{x\to-1} f(x) = \lim_{x\to-1} \sqrt{\frac{(1-x^2)}{(x-2)}} = \sqrt{\frac{1-(-1)^2}{-1-2}} = \sqrt{\frac{0}{-3}} = 0$$

$$\lim_{x\to1} f(x) = \lim_{x\to1} \sqrt{\frac{(1-x^2)}{(x-2)}} = \sqrt{\frac{1-1}{1-2}} = \sqrt{\frac{0}{-1}} = 0$$

$$\lim_{x\to2} f(x) = \lim_{x\to2} \sqrt{\frac{(1-x^2)}{(x-2)}} = +\infty$$

Conclusion,

$$\boxed{E(f) : y \in [0, +\infty) \; is\ the\ range\ of\ a\ function}$$

ANSWER

$$D(f) : x \in (-\infty, -1] \cup [1,2) \; is\ the\ domain\ of\ a\ function$$

$$E(f) : y \in [0, +\infty) \; is\ the\ range\ of\ a\ function$$