Answer on Question #80295 – Math – Calculus

Question

1 Find the integral of $f_1(x) = \frac{1}{\tan^2 x} + \sec^2 x$.

2 Find the integral of $f_2(x) = \frac{1}{\tan^2 x + \sec^2 x}$.

Solution

By definition, $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$.

1. $f_1(x) = \frac{1}{\tan^2 x} + \sec^2 x = \frac{\cos^2 x}{\sin^2 x} + \frac{1}{\cos^2 x}$

$f_1(\cos x, \sin x)$ is rational and equals $f_1(-\cos x, -\sin x) \Longrightarrow$ change of varible $t = \tan x$ is helpful

$\tan x = t, x = \arctan t, dx = \frac{dt}{1+t^2}, \sin^2 x = \frac{t^2}{1+t^2}, \cos^2 x = \frac{1}{1+t^2}$

$I_1 = \int f_1\, dx = \int \frac{1}{t^2}\frac{dt}{1+t^2} + \int \frac{1+t^2}{1}\frac{dt}{1+t^2} = \int \frac{1}{t^2}\frac{dt}{1+t^2} + t = \int \frac{dt}{t^2} - \int \frac{dt}{1+t^2} + t = -\frac{1}{t} - \arctan t + K = -\frac{1}{\tan x} - x + \tan x + K$

$\frac{1}{t^2(1+t^2)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{1+t^2} \Longrightarrow 1 = At(1+t^2) + B(1+t^2) + Ct^3 + Dt^2$

$\Longrightarrow B = 1, A = 0, C = 0, D = -1$

2. $f_2(x) = \frac{1}{\tan^2 x + \sec^2 x} = \frac{\cos^2 x}{\sin^2 x + 1}$

$f_2(\cos x, \sin x)$ is rational and equals $f_2(-\cos x, -\sin x) \Longrightarrow$ change of varible $t = \tan x$ is helpful

$\tan x = t, x = \arctan t, dx = \frac{dt}{1+t^2}, \sin^2 x = \frac{t^2}{1+t^2}, \cos^2 x = \frac{1}{1+t^2}$

$I_2 = \int f_2\, dx = \int \frac{1}{1+2t^2}\frac{dt}{1+t^2} = 2\int \frac{dt}{1+2t^2} - \int \frac{dt}{1+t^2} = \sqrt{2}\arctan(\sqrt{2}t) - \arctan t = -x + \sqrt{2}\arctan(\sqrt{2}\tan x) + K$

$\frac{1}{(1+2t^2)(1+t^2)} = \frac{At+B}{1+2t^2} + \frac{Ct+D}{1+t^2} \Longrightarrow 1 = (At+B)(1+t^2) + (Ct+D)(1+2t^2)$

$\Longrightarrow 1 = (A+2C)t^3 + (B+2D)t^2 + (A+C)t + (B+D)$

$\Longrightarrow C = 0, B = 2, A = 0, D = -1$

Answer:

$\int \left(\frac{1}{\tan^2 x} + \sec^2 x\right) dx = -\frac{1}{\tan x} - x + \tan x + K$

$\int \frac{1}{\tan^2 x + \sec^2 x} dx = -x + \sqrt{2}\arctan(\sqrt{2}\tan x) + K$