Answer on Question #79989 – Math – Algebra

Question

Let x belongs to R such that $0 \le x_1 \le x_2 \le \ldots\ldots \le x_n, n \ge 2$ and $1/(1+x_1) + 1/(1+x_2) + \ldots\ldots + 1/(1+x_n) = 1$, then show that

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \ge (n-1)\left(1/\sqrt{x_1} + \cdots + 1/\sqrt{x_n}\right)$$

Solve using inequalities.

Solution

Let $\dfrac{1}{1+x_i} = a_i, for\ i = 1, 2, \ldots, n.$ Then $\displaystyle\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \frac{1}{1+x_i} = 1.$

We have that

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \ge (n-1)\left(1/\sqrt{x_1} + \cdots + 1/\sqrt{x_n}\right)$$

$$\sqrt{x_i} = \sqrt{\frac{1}{a_i} - 1} = \sqrt{\frac{1-a_i}{a_i}}$$

$$\sqrt{\frac{1-a_i}{a_i}} + \sqrt{\frac{a_i}{1-a_i}} = \sqrt{\frac{(1-a_i)^2}{a_i(1-a_i)}} + \sqrt{\frac{a_i^2}{a_i(1-a_i)}} = \sqrt{\frac{1}{a_i(1-a_i)}}$$

$$\sum_{i=1}^{n} \sqrt{\frac{1-a_i}{a_i}} \ge (n-1) \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} <=>$$

$$<=> \sum_{i=1}^{n} \left(\sqrt{\frac{1-a_i}{a_i}} + \sqrt{\frac{a_i}{1-a_i}} \right) \ge n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} <=>$$

$$<=> \sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}} \ge n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} <=>$$

$$<=> \left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}} \right) \ge n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}}$$

$$n \sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} \le \left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}} \right)$$

The last inequality is true according to Chebyshev's inequality applied to the sequences

$$(a_1, a_2, \dots, a_n) \text{ and } \left(\frac{1}{\sqrt{a_1(1-a_1)}}, \frac{1}{\sqrt{a_2(1-a_2)}}, \dots, \frac{1}{\sqrt{a_n(1-a_n)}}\right)$$

Theorem (Chebyshev's inequality) Let $a_1 \leq a_2 \leq \dots\dots \leq a_n$ and $b_1 \leq b_2 \leq \dots\dots \leq b_n$ be real numbers. Then we have

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \leq n\left(\sum_{i=1}^{n} a_i b_i\right)$$

Note Chebyshev's inequality is also true if $a_1 \geq a_2 \geq \dots\dots \geq a_n$ and $b_1 \geq b_2 \geq \dots\dots \geq b_n$

But if $a_1 \leq a_2 \leq \dots\dots \leq a_n$ and $b_1 \geq b_2 \geq \dots\dots \geq b_n$ (or the reverse) then we have

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq n\left(\sum_{i=1}^{n} a_i b_i\right)$$

$\frac{1}{1+x_i} = a_i, for\ i = 1, 2, \dots, n, 0 \leq x_1 \leq x_2 \leq \dots\dots \leq x_n, n \geq 2,$ then

$$\frac{1}{1+x_1} \geq \frac{1}{1+x_2} \geq \cdots \geq \frac{1}{1+x_n}$$

$$a_1 \geq a_2 \geq \dots\dots \geq a_n$$

The sequence $(a_1, a_2, \dots, a_n)$ is non − increasing.

$$\frac{1}{\sqrt{a_i(1-a_i)}} = \frac{1}{\sqrt{\frac{1}{1+x_i}\left(1-\frac{1}{1+x_i}\right)}} = \frac{1+x_i}{\sqrt{x_i}}$$

$$f(x) = \frac{1+x}{\sqrt{x}} = x^{-1/2} + x^{1/2}, x > 0$$

$$f'(x) = -\frac{1}{2}x^{-3/2} + \frac{1}{3}x^{-1/2} = \frac{x-1}{2x^{3/2}}$$

The function f is increasing on $(1, \infty)$.

$$f\left(\frac{1}{x}\right) = \frac{1+\frac{1}{x}}{\sqrt{\frac{1}{x}}} = \frac{1+x}{\sqrt{x}} = f(x)$$

If $x_1 = 0,$ then $x_2 \geq 0$

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} \leq 1 \Rightarrow \frac{1}{1+0} + \frac{1}{1+x_2} \leq 1 \Rightarrow \frac{1}{1+x_2} \leq 0$$

This is false, if $x_2 \geq 0$.

If $0 < x_1 < 1,$ then

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} \leq 1$$

$$\frac{1}{1+x_2} \le 1 - \frac{1}{1+x_1} = \frac{x_1}{1+x_1} => x_2 \ge \frac{1}{x_1} => x_2 > 1$$

Hence

$$f(x_2) \le \cdots \le f(x_n)$$

We have that

$$f(x_2) \ge f\left(\frac{1}{x_1}\right) = f(x_1)$$

$$f(x_1) \le f(x_2) \le \cdots \le f(x_n)$$

If $x_1 \ge 1,$ then

$$f(x_1) \le f(x_2) \le \cdots \le f(x_n)$$

It is clear that (both in case $x_1 \ge 1$ and in case $x_1 < 1$)

$$f(x_1) \le f(x_2) \le \cdots \le f(x_n)$$

$$\frac{1}{\sqrt{a_1(1-a_1)}} \le \frac{1}{\sqrt{a_2(1-a_2)}} \le \cdots \le \frac{1}{\sqrt{a_n(1-a_n)}}$$

The sequence $\left(\frac{1}{\sqrt{a_1(1-a_1)}}, \frac{1}{\sqrt{a_2(1-a_2)}}, \dots, \frac{1}{\sqrt{a_n(1-a_n)}}\right)$ is $\text{non} - \text{decreasing}.$

Apply the Chebyshev's inequality

$$n\sum_{i=1}^{n} a_i \sqrt{\frac{1}{a_i(1-a_i)}} \le \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}}\right)$$

$$n\sum_{i=1}^{n} \sqrt{\frac{a_i}{1-a_i}} \le \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \sqrt{\frac{1}{a_i(1-a_i)}}\right)$$

Therefore

$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \ge (n-1)\left(1/\sqrt{x_1} + \cdots + 1/\sqrt{x_n}\right).$