ANSWER on Question #79036 – Math – Differential Equations

QUESTION

Find the integral surface of the PDE

$$(x-y)p+(y-x-z)q=z$$

SOLUTION

Consider a quasilinear equation

$$a(x,y,z)p+b(x,y,z)q=c(x,y,z).$$

By Lagrange's method the auxiliary equations are as following:

$$\frac{dx}{a(x,y,z)}=\frac{dy}{b(x,y,z)}=\frac{dz}{c(x,y,z)}.$$

So, for the given quasilinear equation we come to the system in the symmetric form

$$\frac{dx}{x-y}=\frac{dy}{y-x-z}=\frac{dz}{z}. \qquad (1)$$

One of a way to solve the system in symmetric form is to use the equal fractions property

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}=\frac{\lambda_1 a_1+\lambda_2 a_2+\cdots+\lambda_n a_n}{\lambda_1 b_1+\lambda_2 b_2+\cdots+\lambda_n b_n}$$

In our case,

Choosing $\lambda_1=\lambda_2=\lambda_3=1$ as multipliers, each fraction on (1):

$$\frac{dx}{x-y}=\frac{dy}{y-x-z}=\frac{dz}{z}=\frac{1\cdot dx+1\cdot dy+1\cdot dz}{1\cdot(x-y)+1\cdot(y-x-z)+1\cdot z}\to$$

$$\frac{dx}{x-y}=\frac{dy}{y-x-z}=\frac{dz}{z}=\frac{d(x+y+z)}{x-y+y-x-z+z}\to\frac{dx}{x-y}=\frac{dy}{y-x-z}=\frac{dz}{z}=\frac{d(x+y+z)}{0}\to$$

$$d(x+y+z)=0\to\boxed{x+y+z=C_1} \qquad (2)$$

Take the last two fractions of (1) and using (2) we get

$$\begin{cases}\dfrac{dy}{y-x-z}=\dfrac{dz}{z}\\ x+y+z=C_1\end{cases}\rightarrow\begin{cases}\dfrac{dy}{y-(x+z)}=\dfrac{dz}{z}\\ x+z=C_1-y\end{cases}\rightarrow\frac{dy}{y-(C_1-y)}=\frac{dz}{z}\rightarrow\frac{dy}{y-C_1+y}=\frac{dz}{z}\rightarrow$$

$$\frac{dy}{2y-C_1}=\frac{dz}{z}\rightarrow\int\frac{dy}{y-C_1+y}=\int\frac{dz}{z}\rightarrow\frac{1}{2}\cdot\ln|2y-C_1|=\ln|z|+\ln|C_2|\Big|\times(2)\rightarrow$$

$$\ln|2y-C_1|=2\cdot\ln|z|+\underbrace{2\cdot\ln|C_2|}_{\ln|C_3|}\rightarrow\ln|2y-C_1|-\ln|z^2|=\ln|C_3|\rightarrow\ln\left|\frac{2y-C_1}{z^2}\right|=\ln|C_3|\rightarrow$$

$$\frac{(2y-C_1)}{z^2}=C_3\rightarrow\begin{bmatrix}Again, we\ use\ equality\ (2)\\ x+y+z=C_1\end{bmatrix}\rightarrow\frac{\big(2y-(x+y+z)\big)}{z^2}=C_3\rightarrow$$

$$\frac{2y-x-y-z}{z^2}=C_3\rightarrow\boxed{\frac{y-x-z}{z^2}=C_3}$$

We have found two integrals for the given equation

$$\begin{cases}x+y+z=C_1\\ \dfrac{y-x-z}{z^2}=C_3\end{cases}$$

Therefore, any integral surface of the differential equation $(x-y)p+(y-x-z)q=z$ is described by the equation

$$\varphi(C_1,C_3)=0\rightarrow\boxed{\varphi\left(x+y+z,\frac{y-x-z}{z^2}\right)=0}$$

where φ is an arbitrary function, a smooth function.

ANSWER: $(x-y)p+(y-x-z)q=z\rightarrow\varphi\left(x+y+z,\frac{y-x-z}{z^2}\right)=0.$