Answer on Question #79020 – Math – Calculus
Question

Prove that

$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2},$

where $I_n = \int \sec^n x \, dx$

Hence deduce I_4.

Solution

$$I_n = \int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$$

$\int u \, dv = uv - \int v \, du$

$u = \sec^{n-2} x \, , du = (n-2) \sec^{n-3} x \left(-\frac{-\sin x}{\cos^2 x} \right) dx =$

$= (n-2) \sec^{n-3} x \sec x \tan x \, dx = (n-2) \sec^{n-2} x \tan x \, dx$

$dv = \sec^2 x \, dx, v = \int \sec^2 x \, dx = \tan x$

$I_n = \int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx =$

$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x \tan^2 x \, dx$

$1 + \tan^2 x = \sec^2 x => \tan^2 x = \sec^2 x - 1$

$I_n = \int \sec^n x \, dx = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x \tan^2 x \, dx =$

$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-2} x (\sec^2 x - 1) \, dx =$

$= \sec^{n-2} x \tan x - \int (n-2) \sec^n x \, dx + \int (n-2) \sec^{n-2} x \, dx =$

$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

Solve for I_n

$I_n (1 + n - 2) = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

$I_4 = \frac{\sec^{4-2} x \tan x}{4-1} + \frac{4-2}{4-1} I_{4-2}$

$I_{4-2} = I_2 = \int \sec^2 x \, dx = \tan x + C_1$

$$I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x + C$$