Answer on Question #78993 – Math – Calculus

Question

Prove that $\frac{x}{1+x^2} < \arctan(x) < x$ for every $x > 0$.

Proof

It is enough to show that $\frac{x}{1+x^2} < \arctan(x)$ and $\arctan(x) < x$ for every $x > 0$.

1) Consider the function $f(x) = \arctan(x) - \frac{x}{1+x^2},\ x \geq 0$. Note that f is continuous on $(0; +\infty)$ and for every $x > 0$ there exists $f'(x)$.
   Since $f'(x) = (\arctan(x))' - \left(\frac{x}{1+x^2}\right)' = \frac{1}{1+x^2} - \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0$ for every $x > 0$ and $f'(0) = 0$, from the monotonicity criterion, it follows that function f is strictly increasing on $[0; +\infty)$. It is also obvious that $f(0) = 0$.
   This implies that $f(x) > f(0) = 0$ for every $x > 0$ which is equivalent to the next inequality:

$$\arctan(x) - \frac{x}{1+x^2} > 0, x > 0 \leftrightarrow \arctan(x) > \frac{x}{1+x^2}, x > 0.$$

2) Consider the function $g(x) = x - \arctan(x),\ x \geq 0$. Note that g is continuous on $(0; +\infty)$ and for every $x > 0$ there exists $g'(x)$.
   Since $g'(x) = (\mathrm{x})' - (\arctan(x))' = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0$ for every $x > 0$ and $g'(0) = 0$, from the monotonicity criterion, it follows that function g is strictly increasing on $[0; +\infty)$. It is also obvious that $g(0) = 0$.
   This implies that $g(x) > g(0) = 0$ for every $x > 0$ which is equivalent to the next inequality:

$$x - \arctan(x) > 0, x > 0 \leftrightarrow x > \arctan(x), x > 0.$$

From 1) and 2) we have the required statement.