ANSWER on Question #78967 – Math – Calculus

QUESTION

Prove that

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos^n x}}{\sqrt{\cos^n x}+\sqrt{\sin^n x}}dx=\frac{\pi}{4}, \quad \forall n \in \mathbb{N}$$

SOLUTION

We recall the property of a definite integral, the binomial and trigonometric formulas that we will need to solve this question

$$if\ f(x)-odd\ function, then \int_{-a}^{a} f(x)dx=0$$

(More information: https://en.wikipedia.org/wiki/Riemann_integral)

$$(a+b)^2=a^2+2ab+b^2$$

$$(a-b)(a+b)=a^2-b^2$$

(More information: https://en.wikipedia.org/wiki/Binomial_theorem)

$$\tan x=\frac{\sin x}{\cos x}\to \tan(-x)=-\tan(x)$$

$$\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+1}{\tan x-1}$$

(More information: https://en.wikipedia.org/wiki/List_of_trigonometric_identities)

Then,

1 STEP: We transform the integrand

$$\frac{\sqrt{\cos^n x}}{\sqrt{\cos^n x}+\sqrt{\sin^n x}}=\frac{2\sqrt{\cos^n x}}{2\cdot\left(\sqrt{\cos^n x}+\sqrt{\sin^n x}\right)}=\frac{\left(\sqrt{\cos^n x}+\sqrt{\sin^n x}\right)+\left(\sqrt{\cos^n x}-\sqrt{\sin^n x}\right)}{2\cdot\left(\sqrt{\cos^n x}+\sqrt{\sin^n x}\right)}=$$

$$=\frac{\left(\sqrt{\cos^n x}+\sqrt{\sin^n x}\right)}{2\cdot\left(\sqrt{\cos^n x}+\sqrt{\sin^n x}\right)}+\frac{\left(\sqrt{\cos^n x}-\sqrt{\sin^n x}\right)}{2\cdot\left(\sqrt{\cos^n x}+\sqrt{\sin^n x}\right)}=\frac{1}{2}+\frac{\left(\sqrt{\cos^n x}-\sqrt{\sin^n x}\right)}{2\cdot\left(\sqrt{\cos^n x}+\sqrt{\sin^n x}\right)}=$$

$$=\frac{1}{2}+\frac{1}{2}\cdot\frac{\sqrt{\cos^n x}\cdot\left(1-\frac{\sqrt{\sin^n x}}{\sqrt{\cos^n x}}\right)}{\sqrt{\cos^n x}\cdot\left(1+\frac{\sqrt{\sin^n x}}{\sqrt{\cos^n x}}\right)}=\frac{1}{2}+\frac{1}{2}\cdot\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}$$

Conclusion,

$$\boxed{\frac{\sqrt{\cos^n x}}{\sqrt{\cos^n x}+\sqrt{\sin^n x}}=\frac{1}{2}+\frac{1}{2}\cdot\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}}$$

2 STEP: Apply our transformed function

$$\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cos^n x}}{\sqrt{\cos^n x}+\sqrt{\sin^n x}}dx=\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}\right)dx=\int_0^{\frac{\pi}{2}}\frac{1}{2}dx+\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\cdot\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}\right)dx=$$

$$=\frac{x}{2}\Big|_0^{\frac{\pi}{2}}+\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}\left(\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}\right)dx=\frac{1}{2}\cdot\left(\frac{\pi}{2}-0\right)+\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}\left(\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}\right)dx=$$

$$=\frac{\pi}{4}+\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}\left(\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}\right)dx$$

Conclusion,

$$\boxed{\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cos^n x}}{\sqrt{\cos^n x}+\sqrt{\sin^n x}}dx=\frac{\pi}{4}+\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}\left(\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}\right)dx}$$

It remains to prove that

$$\int_0^{\frac{\pi}{2}}\left(\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}\right)dx\equiv 0$$

Then,

$$\int_0^{\frac{\pi}{2}}\left(\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}\right)dx=\begin{bmatrix}x-\frac{\pi}{4}=t\to dx=dt\\ x=t+\frac{\pi}{4}\\ x=\frac{\pi}{2}\to t=\frac{\pi}{4}\\ x=0\to t=-\frac{\pi}{4}\end{bmatrix}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{1-\sqrt{\tan^n\left(t+\frac{\pi}{4}\right)}}{1+\sqrt{\tan^n\left(t+\frac{\pi}{4}\right)}}\right)dt$$

3 STEP: We transform the integrand

$$\frac{1-\sqrt{\tan^n\left(t+\frac{\pi}{4}\right)}}{1+\sqrt{\tan^n\left(t+\frac{\pi}{4}\right)}} \equiv \frac{1-\left(\frac{1+\tan t}{1-\tan t}\right)^{\frac{n}{2}}}{1+\left(\frac{1+\tan t}{1-\tan t}\right)^{\frac{n}{2}}} = \frac{\frac{(1-\tan t)^{\frac{n}{2}}-(1+\tan t)^{\frac{n}{2}}}{(1-\tan t)^{\frac{n}{2}}}}{\frac{(1-\tan t)^{\frac{n}{2}}+(1+\tan t)^{\frac{n}{2}}}{(1-\tan t)^{\frac{n}{2}}}} =$$

$$= \frac{(1-\tan t)^{\frac{n}{2}}-(1+\tan t)^{\frac{n}{2}}}{(1-\tan t)^{\frac{n}{2}}+(1+\tan t)^{\frac{n}{2}}} = \frac{\left[(1-\tan t)^{\frac{n}{2}}-(1+\tan t)^{\frac{n}{2}}\right]\cdot\left[(1-\tan t)^{\frac{n}{2}}+(1+\tan t)^{\frac{n}{2}}\right]}{\left[(1-\tan t)^{\frac{n}{2}}+(1+\tan t)^{\frac{n}{2}}\right]\cdot\left[(1-\tan t)^{\frac{n}{2}}+(1+\tan t)^{\frac{n}{2}}\right]} =$$

$$= \frac{\left((1-\tan t)^{\frac{n}{2}}\right)^2-\left((1+\tan t)^{\frac{n}{2}}\right)^2}{\left((1-\tan t)^{\frac{n}{2}}\right)^2+2\cdot(1-\tan t)^{\frac{n}{2}}\cdot(1+\tan t)^{\frac{n}{2}}+\left((1+\tan t)^{\frac{n}{2}}\right)^2} =$$

$$= \frac{(1-\tan t)^n-(1+\tan t)^n}{(1-\tan t)^n+2\cdot(1-\tan^2 t)^{\frac{n}{2}}+(1+\tan t)^n}$$

Conclusion,

$$\boxed{\frac{1-\sqrt{\tan^n\left(t+\frac{\pi}{4}\right)}}{1+\sqrt{\tan^n\left(t+\frac{\pi}{4}\right)}} = \frac{(1-\tan t)^n-(1+\tan t)^n}{(1-\tan t)^n+2\cdot(1-\tan^2 t)^{\frac{n}{2}}+(1+\tan t)^n}}$$

Then,

$$\int_0^{\frac{\pi}{2}}\left(\frac{1-\sqrt{\tan^n x}}{1+\sqrt{\tan^n x}}\right)dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{(1-\tan t)^n-(1+\tan t)^n}{(1-\tan t)^n+2\cdot(1-\tan^2 t)^{\frac{n}{2}}+(1+\tan t)^n}\right)dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f_n(t)dt$$

4 STEP: Let us prove that the function $f_n(t)$ is odd

$$f_n(t) = \frac{(1-\tan t)^n-(1+\tan t)^n}{(1-\tan t)^n+2\cdot(1-\tan^2 t)^{\frac{n}{2}}+(1+\tan t)^n}$$

Then,

$$f_n(-t) = \frac{(1-\tan(-t))^n-(1+\tan(-t))^n}{(1-\tan(-t))^n+2\cdot(1-\tan^2(-t))^{\frac{n}{2}}+(1+\tan(-t))^n} =$$

$$= \frac{(1+\tan t)^n-(1-\tan t)^n}{(1+\tan t)^n+2\cdot(1-\tan^2 t)^{\frac{n}{2}}+(1-\tan t)^n} = \frac{(-1)\cdot((1-\tan t)^n-(1+\tan t)^n)}{(1+\tan t)^n+2\cdot(1-\tan^2 t)^{\frac{n}{2}}+(1-\tan t)^n} =$$

$$= (-1) \cdot f_n(t)$$

Conclusion,

$$\boxed{f_n(-t) = (-1) \cdot f_n(t) \rightarrow f_n(t) - odd\ function}$$

Then,

$$\boxed{\int_0^{\frac{\pi}{2}} \left(\frac{1 - \sqrt{\tan^n x}}{1 + \sqrt{\tan^n x}}\right) dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f_n(t) dt \equiv 0, \quad \forall n \in \mathbb{N}}$$

General conclusion,

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos^n x}}{\sqrt{\cos^n x} + \sqrt{\sin^n x}} dx = \frac{\pi}{4} + \frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} \left(\frac{1 - \sqrt{\tan^n x}}{1 + \sqrt{\tan^n x}}\right) dx = \frac{\pi}{4} + \frac{1}{2} \cdot 0 = \frac{\pi}{4} \rightarrow$$

$$\boxed{\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos^n x}}{\sqrt{\cos^n x} + \sqrt{\sin^n x}} dx = \frac{\pi}{4}, \quad \forall n \in \mathbb{N}}$$

Q.E.D.