Answer on Question #78965 – Math – Calculus

Question

Given an example of a function which is not integrable on [0, π]. Justify your answer.

Solution

Consider

$$f(x) = \begin{cases} 0, x \in \mathbb{Q}, \\ 1, x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ is the set of all rational numbers.

Then for any partition

$$0 = x_0 < x_1 < x_2 < \ldots < x_n = \pi$$

we may obtain integral sum

$\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} 0 \cdot (x_{i+1} - x_i) = 0$ (if we take all rational t_i)

or

$\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \neq 0$ (if we take all irrational t_i)

This means that the limit of such sums doesn't exist because the limit of integral sum will depend on the choice of partition. The function $f(x)$ is not integrable.

Answer: $f(x) = \begin{cases} 0, x \in \mathbb{Q}, \\ 1, x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$