Answer on Question #78899 – Math – Differential Equations

Question

$$y' + (\cot x)y = x\csc x\,, y\left(\frac{\pi}{2}\right) = 1$$

$$y = -\frac{x^2}{2\sin x} - \frac{1-\frac{\pi^2}{8}}{\sin x}$$

$$y = \frac{x^2}{2\sin x} + \frac{1-\frac{\pi^2}{8}}{\sin x}$$

$$y = -\frac{x^2}{2\sin x} + \frac{1+\frac{\pi^2}{8}}{\sin x}$$

$$y = -\frac{x^2}{2\sin x} - \frac{1-\frac{\pi^2}{8}}{\sin x}$$

Solution

We have a linear first order differential equation

$$\frac{dy}{dx} + (\cot x)y = x\csc x$$

$P(x) = \cot x\,, Q(x) = x\csc x$

Integrating factor

$IF = e^{\int P(x)dx} = e^{\int \cot x dx}$

Indefinite integral

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx$$

Substitution

$u = \sin x\,, du = \cos x\,dx$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx = \int \frac{1}{u}du = \ln|u| + C_1 = \ln|\sin x| + C_1$$

$IF = e^{\int P(x)dx} = e^{\int \cot x dx} = e^{\ln(\sin x)} = \sin x$

$$\sin x\frac{dy}{dx} + \sin x\,(\cot x)y = x\sin x\csc x$$

$$\frac{d}{dx}(y\sin x) = x\sin x\csc x$$

$$\frac{d}{dx}(y\sin x) = x$$

Integrate

$$y\sin x = \frac{x^2}{2} + C$$

$$y\left(\frac{\pi}{2}\right) = 1 => 1\left(\sin\frac{\pi}{2}\right) = \frac{\left(\frac{\pi}{2}\right)^2}{2} + C => C = 1 - \frac{\pi^2}{8}$$

$$y\sin x = \frac{x^2}{2} + 1 - \frac{\pi^2}{8}$$

$$y = \frac{x^2}{2\sin x} + \frac{1 - \frac{\pi^2}{8}}{\sin x}$$

Answer: $y = \frac{x^2}{2\sin x} + \frac{1 - \frac{\pi^2}{8}}{\sin x}$.