ANSWER on Question #78896 – Math – Differential Equations

QUESTION

Obtain a solution of the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 16 \cdot \frac{\partial^2 u(x,t)}{\partial x^2}$$

for $0 \leq x \leq \pi$ and $t > 0$ and the following boundary and initial conditions:

$$\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases} - boundary\ conditions$$

$$\begin{cases} u(x,0) = x(\pi - x) \\ \dfrac{\partial u(x,0)}{\partial t} = 0 \end{cases} - initial\ conditions$$

SOLUTION

0 STEP: separation of variables.

Let

$$u(x,t) = X(x)T(t) \to \begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial^2}{\partial t^2}\big(X(x)T(t)\big) = X(x) \cdot \dfrac{d^2\big(T(t)\big)}{dt^2} = X(x) \cdot T''(t) \\ \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial^2}{\partial x^2}\big(X(x)T(t)\big) = T(t) \cdot \dfrac{d^2\big(X(x)\big)}{dx^2} = X''(x) \cdot T(t) \end{cases}$$

Boundary conditions:

$$\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases} \to \begin{cases} u(0,t) = X(0)T(t) = 0, \forall t > 0 \\ u(\pi,t) = X(\pi)T(t) = 0, \forall t > 0 \end{cases} \to \boxed{\begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}}$$

Then,

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 16 \cdot \frac{\partial^2 u(x,t)}{\partial x^2} \to X(x) \cdot T''(t) = 16 \cdot X''(x) \cdot T(t) \Big| \times \frac{1}{16X(x)T(t)} \to$$

$$\frac{X(x) \cdot T''(t)}{16X(x)T(t)} = \frac{16 \cdot X''(x) \cdot T(t)}{16X(x)T(t)} \to \boxed{\frac{1}{16} \cdot \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda}$$

1 STEP: We solve the Sturm-Liouville problem.

(More information: https://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville_theory)

In our case,

$$\begin{cases} \dfrac{X''(x)}{X(x)} = -\lambda \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

$$\frac{X''(x)}{X(x)} = -\lambda \rightarrow X''(x) = -\lambda X(x) \rightarrow X''(x) + \lambda X(x) = 0$$

Let us find the solutions of the given equation in the form

$$X(x) = e^{kx} \rightarrow X''(x) = k^2 \cdot e^{kx}$$

Then,

$$X''(x) + \lambda X(x) = 0 \rightarrow k^2 \cdot e^{kx} + \lambda e^{kx} = 0 \rightarrow e^{kx}(k^2 + \lambda) = 0 \rightarrow k^2 = -\lambda$$

$$k^2 = -\lambda \rightarrow \left[\begin{array}{l} k_1 = \sqrt{-\lambda} = i\sqrt{\lambda} \\ k_2 = -\sqrt{-\lambda} = -i\sqrt{\lambda} \end{array} \right.$$

Then,

$$X(x) = C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x} \equiv A_1 \cos(\sqrt{\lambda}x) + A_2 \sin(\sqrt{\lambda}x)$$

$$\boxed{X(x) = A_1 \cos(\sqrt{\lambda}x) + A_2 \sin(\sqrt{\lambda}x)}$$

$$X(0) = 0 = A_1 \cos(\sqrt{\lambda} \cdot 0) + A_2 \sin(\sqrt{\lambda} \cdot 0) = A_1 \cos(0) + A_2 \sin(0) = A_1 \cdot 1 + A_2 \cdot 0 \rightarrow$$

$$\boxed{A_1 = 0}$$

$$X(\pi) = 0 = A_2 \sin(\sqrt{\lambda}\pi) \rightarrow \sin(\sqrt{\lambda}\pi) = 0 \rightarrow \sqrt{\lambda}\pi = \pi n, n = 1,2,3, \ldots$$

$$\boxed{\lambda_n = n^2, n = 1,2,3, \ldots}$$

Conclusion,

$$\boxed{\begin{cases} X_n(x) = A \cdot \sin(nx) \\ \lambda_n = n^2 \\ n = 1,2,3, \ldots \end{cases}}$$

2 STEP: Finding the general solution.

$$\frac{1}{16}\cdot\frac{T''(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda_n\rightarrow\frac{1}{16}\cdot\frac{T''(t)}{T(t)}=-n^2\rightarrow T''(t)=-16n^2T(t)\rightarrow$$

$$T''(t)+16n^2T(t)=0$$

Let us find the solutions of the given equation in the form

$$T(t)=e^{kt}\rightarrow T''(t)=k^2\cdot e^{kt}$$

Then,

$$T''(t)+16n^2T(t)=0\rightarrow k^2\cdot e^{kt}+16n^2e^{kt}=0\rightarrow e^{kt}(k^2+16n^2)=0\rightarrow k^2=-16n^2$$

$$k^2=-16n^2\rightarrow\left[\begin{array}{l}k_1=\sqrt{-16n^2}=4in\\k_2=-\sqrt{-16n^2}=-4in\end{array}\right.$$

Then,

$$T_n(x)=C_1e^{4int}+C_2e^{-4int}\equiv A_1\cos(4nt)+A_2\sin(4nt)$$

$$\boxed{T_n(x)=A_n^{(1)}\cos(4nt)+A_n^{(1)}\sin(4nt)}$$

Then,

$$u_n(x,t)=X_n(x)\cdot T_n(t)=(A\cdot\sin(nx))\cdot\left(A_n^{(1)}\cos(4nt)+A_n^{(1)}\sin(4nt)\right)\rightarrow$$

$$\boxed{u_n(x,t)=\left(B_n^{(1)}\cos(4nt)+B_n^{(2)}\sin(4nt)\right)\sin(nx)-particular\ solution}$$

Conclusion,

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)-general\ solution$$

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}\left(B_n^{(1)}\cos(4nt)+B_n^{(2)}\sin(4nt)\right)\sin(nx)}$$

3 STEP: Determine the coefficients $B_n^{(1)}$ and $B_n^{(2)}$.

To do this, you must use the initial conditions.

$$\frac{\partial u(x,0)}{\partial t} = 0 \rightarrow$$

$$\frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty}\left(B_n^{(1)}\cos(4nt) + B_n^{(2)}\sin(4nt)\right)\sin(nx)\right)_{t=0} =$$

$$= \left(\sum_{n=1}^{\infty}\left(-4nB_n^{(1)}\sin(4nt) + 4nB_n^{(2)}\cos(4nt)\right)\sin(nx)\right)_{t=0} =$$

$$= \sum_{n=1}^{\infty}\left(-4nB_n^{(1)}\sin(4n\cdot 0) + 4nB_n^{(2)}\cos(4n\cdot 0)\right)\sin(nx) =$$

$$= \sum_{n=1}^{\infty}\left(-4nB_n^{(1)}\cdot 0 + 4nB_n^{(2)}\cdot 1\right)\sin(nx) = \sum_{n=1}^{\infty} 4nB_n^{(2)}\sin(nx) = 0$$

Then,

$$4nB_n^{(2)} = 0 \rightarrow \boxed{B_n^{(2)} = 0}$$

Conclusion,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n^{(1)}\cos(4nt)\sin(nx)}$$

$$u(x,0) = x(\pi - x) \rightarrow$$

$$u(x,0) = \left(\sum_{n=1}^{\infty} B_n^{(1)} \cos(4nt)\sin(nx)\right)_{t=0} = \sum_{n=1}^{\infty} B_n^{(1)} \cos(4n \cdot 0)\sin(nx) =$$

$$= \sum_{n=1}^{\infty} B_n^{(1)} \cdot 1 \cdot \sin(nx) \rightarrow \sum_{n=1}^{\infty} B_n^{(1)} \sin(nx) = x(\pi - x)$$

As we know

$$\int_0^{\pi} \sin(mx) \cdot \sin(nx)\, dx = \begin{cases} \frac{\pi}{2}, n = m \\ 0, n \neq m \end{cases}$$

In our case,

$$\int_0^{\pi} \times \left| \sum_{n=1}^{\infty} B_n^{(1)} \sin(nx) = x(\pi - x) \right| \times \sin(mx)\, dx$$

$$B_m^{(1)} = \int_0^{\pi} x(\pi - x)\sin(mx)\, dx = \int_0^{\pi} (\pi x - x^2)\sin(mx)\, dx \rightarrow$$

$$B_m^{(1)} = \int_0^{\pi} \pi x \sin(mx)\, dx - \int_0^{\pi} x^2 \sin(mx)\, dx = I_1 - I_2$$

$$I_1=\int_0^{\pi}\pi x\sin(mx)\,dx=\pi\cdot\int_0^{\pi}\underbrace{x}_{u}\cdot\underbrace{\sin(mx)\,dx}_{dv}=\begin{bmatrix}u=x\to du=dx\\ dv=\sin(mx)\,dx\\ v=\dfrac{-\cos(mx)}{m}\end{bmatrix}=$$

$$=\pi\cdot\left(-\frac{x\cdot\cos(mx)}{m}\Bigg|_0^{\pi}-\int_0^{\pi}\frac{-\cos(mx)\,dx}{m}\right)=$$

$$=\pi\cdot\left(-\frac{\pi\cdot\cos(\pi m)}{m}-\left(-\frac{0\cdot\cos(m\cdot 0)}{m}\right)+\frac{1}{m}\int_0^{\pi}\cos(mx)\,dx\right)=$$

$$=\pi\cdot\left(-\frac{\pi\cdot(-1)^m}{m}+\frac{1}{m}\cdot\frac{\sin(mx)}{m}\Bigg|_0^{\pi}\right)=\pi\cdot\left(-\frac{\pi\cdot(-1)^m}{m}+\frac{\sin(m\cdot\pi)}{m^2}-\frac{\sin(m\cdot 0)}{m^2}\right)=$$

$$=\pi\cdot\left(-\frac{\pi\cdot(-1)^m}{m}+0-0\right)=-\frac{\pi^2\cdot(-1)^m}{m}$$

Conclusion,

$$\boxed{I_1=-\frac{\pi^2\cdot(-1)^m}{m}}$$

$$I_2=\int_0^{\pi}\underbrace{x^2}_{u}\cdot\underbrace{\sin(mx)\,dx}_{dv}=\begin{bmatrix}u=x^2\to du=2xdx\\ dv=\sin(mx)\,dx\\ v=\dfrac{-\cos(mx)}{m}\end{bmatrix}=$$

$$=-\frac{x^2\cdot\cos(mx)}{m}\Bigg|_0^{\pi}-\int_0^{\pi}\frac{-2x\cos(mx)\,dx}{m}=$$

$$=-\frac{\pi^2\cdot\cos(m\pi)}{m}-\left(-\frac{0^2\cdot\cos(m\cdot 0)}{m}\right)+\frac{2}{m}\cdot\int_0^{\pi}x\cos(mx)\,dx=$$

$$=-\frac{\pi^2\cdot(-1)^m}{m}+\frac{2}{m}\cdot\int_0^{\pi}\underbrace{x}_{u}\cdot\underbrace{\cos(mx)\,dx}_{dv}=\begin{bmatrix}u=x\to du=dx\\ dv=\cos(mx)\,dx\\ v=\dfrac{\sin(mx)}{m}\end{bmatrix}=$$

$$=-\frac{\pi^2\cdot(-1)^m}{m}+\frac{2}{m}\cdot\left(\frac{x\cdot\sin(mx)}{m}\Bigg|_0^{\pi}-\int_0^{\pi}\frac{\sin(mx)}{m}dx\right)=$$

$$= -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m} \cdot \left(\frac{\pi \cdot \sin(m\pi)}{m} - \frac{0 \cdot \sin(m \cdot 0)}{m} - \frac{1}{m} \int_0^{\pi} \sin(mx)\, dx \right) =$$

$$= -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m} \cdot \left(0 - 0 - \frac{1}{m} \cdot \left(-\frac{\cos(mx)}{m} \Big|_0^{\pi} \right) \right) =$$

$$= -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m^3} \cdot (\cos(m\pi) - \cos(m \cdot 0)) =$$

$$= -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m^3} \cdot ((-1)^m - 1)$$

$$\boxed{I_2 = -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m^3} \cdot ((-1)^m - 1)}$$

Then,

$$B_m^{(1)} = I_1 - I_2 = -\frac{\pi^2 \cdot (-1)^m}{m} - \left(-\frac{\pi^2 \cdot (-1)^m}{m} + \frac{2}{m^3} \cdot ((-1)^m - 1) \right) \rightarrow$$

$$B_m^{(1)} = -\frac{\pi^2 \cdot (-1)^m}{m} + \frac{\pi^2 \cdot (-1)^m}{m} - \frac{2}{m^3} \cdot ((-1)^m - 1) \rightarrow$$

$$\boxed{B_m^{(1)} = \frac{-2 \cdot ((-1)^m - 1)}{m^3}}$$

As we know

$$\begin{cases} (-1)^m - 1 = 0, m = 2k, k = 1,2,3,4, \dots \\ (-1)^m - 1 = -2, m = 2k-1, k = 1,2,3,4, \dots \end{cases} \rightarrow$$

$$\begin{cases} B_m^{(1)} = 0, m = 2k, k = 1,2,3,4, \dots \\ B_m^{(1)} = \frac{4}{m^3}, m = 2k-1, k = 1,2,3,4, \dots \end{cases}$$

Conclusion,

$$u(x,t) = \sum_{n=1}^{\infty} B_n^{(1)} \cos(4nt) \sin(nx) = \sum_{n=1}^{\infty} \left(\frac{-2 \cdot ((-1)^n - 1)}{n^3} \right) \cos(4nt) \sin(nx)$$

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} \left(\frac{4}{(2k-1)^3} \right) \cos(4(2k-1)t) \sin\big((2k-1)x\big)}$$

ANSWER:

$$u(x,t) = \sum_{k=1}^{\infty} \left(\frac{4}{(2k-1)^3}\right) \cos(4(2k-1)t) \sin\big((2k-1)x\big)$$