Answer on Question #78895 – Math – Calculus

Question

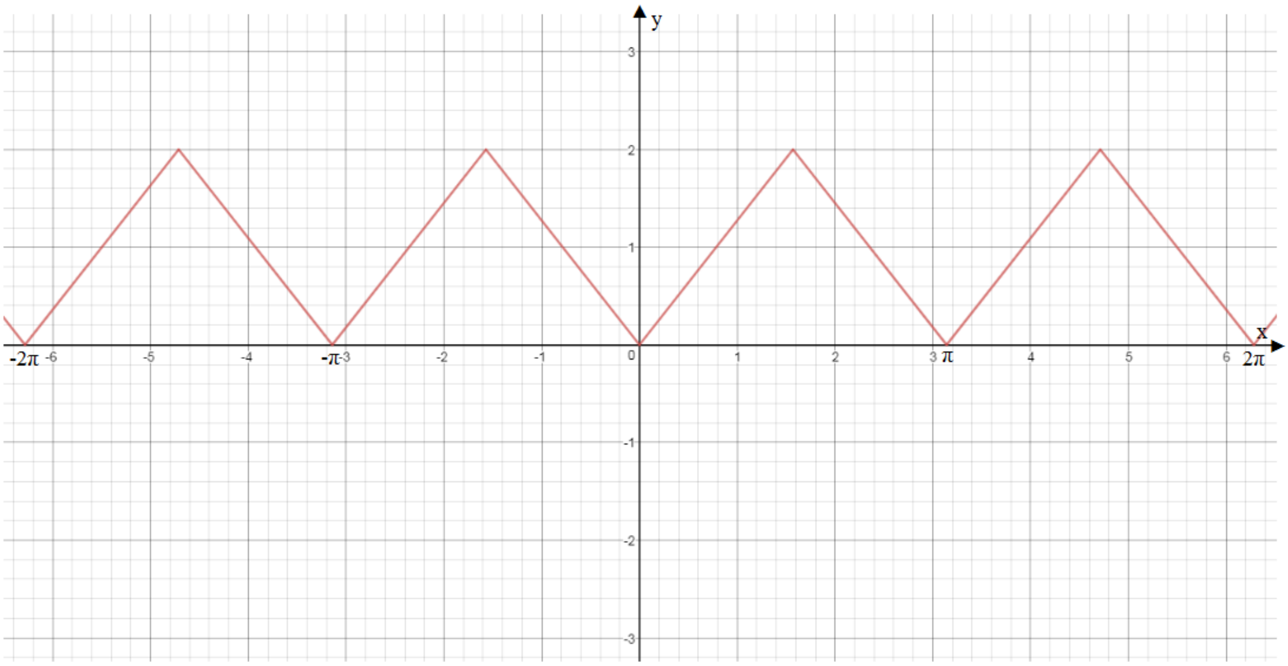
Obtain the Fourier series expansion for the following periodic function which has a period of π

$$f(x)=\begin{cases}\frac{4}{\pi}x & for\ 0\le x\le\frac{\pi}{2}\\ -\frac{4}{\pi}x & for -\frac{\pi}{2}\le x\le 0\end{cases}$$

Solution

$$f(x)=\begin{cases}\frac{4}{\pi}x & for\ 0\le x\le\frac{\pi}{2}\\ -\frac{4}{\pi}x & for -\frac{\pi}{2}\le x\le 0\end{cases}$$

$$f(x+\pi)=f(x)$$



$$a_0=\frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2}f(x)\,dx=\frac{2}{\pi}\left(\int_{-\pi/2}^{0}\left(-\frac{4}{\pi}x\right)dx+\int_{0}^{\pi/2}\left(\frac{4}{\pi}x\right)dx\right)=$$

$$=\frac{2}{\pi}\left(\frac{4}{\pi}\right)\left(\left[-\frac{x^2}{2}\right]_{-\pi/2}^{0}+\left[\frac{x^2}{2}\right]_{0}^{\pi/2}\right)=$$

$$=\frac{8}{\pi}\left(-\left(0-\frac{(-\pi/2)^2}{2}\right)+\frac{(-\pi/2)^2}{2}-0\right)=2$$

$$a_n=\frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2}f(x)\cos\left(n\frac{\pi}{\pi/2}x\right)dx=$$

$$= \frac{2}{\pi}\left(\int_{-\pi/2}^{0}\left(-\frac{4}{\pi}x\right)\cos(2nx)\,dx + \int_{0}^{\pi/2}\left(\frac{4}{\pi}x\right)\cos(2nx)\,dx\right)$$

Indefinite integral

$$\int x\cos(2nx)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$$u = x, du = dx$$

$$dv = \cos(2nx)\,dx, v = \int \cos(2nx)\,dx = \frac{1}{2n}\sin(2nx)$$

$$\int x\cos(2nx)\,dx = \frac{1}{2n}x\sin(2nx) - \frac{1}{2n}\int \sin(2nx)\,dx =$$

$$= \frac{1}{2n}x\sin(2nx) + \frac{1}{4n^2}\cos(2nx) + C$$

$$a_n = \frac{8}{\pi^2}\left(-\left[\frac{1}{2n}x\sin(2nx) + \frac{1}{4n^2}\cos(2nx)\right]_{-\pi/2}^{0}\right) +$$

$$+\frac{8}{\pi^2}\left(\left[\frac{1}{2n}x\sin(2nx) + \frac{1}{4n^2}\cos(2nx)\right]_{0}^{\pi/2}\right) =$$

$$= \frac{8}{\pi^2}\left(-\left(0 + \frac{1}{4n^2}\cos(0) - \left(0 + \frac{1}{4n^2}\cos(-\pi n)\right)\right)\right) +$$

$$+\frac{8}{\pi^2}\left(\left(0 + \frac{1}{4n^2}\cos(\pi n) - \left(0 + \frac{1}{4n^2}\cos(0)\right)\right)\right) =$$

$$= \frac{8}{\pi^2}\left(-\frac{1}{4n^2} + \frac{1}{4n^2}(-1)^n + \frac{1}{4n^2}(-1)^n - \frac{1}{4n^2}\right) = -\frac{8}{\pi^2(2k+1)^2}\ , k = 0, 1, 2, \ldots$$

Since $f(x)$ is odd function, then

$$b_n = \frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2} f(x)\sin\left(n\frac{\pi}{\pi/2}x\right)dx = 0$$

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$$= \frac{2}{\pi}\left(\int_{-\pi/2}^{0}\left(-\frac{4}{\pi}x\right)\sin(2nx)\,dx + \int_{0}^{\pi/2}\left(\frac{4}{\pi}x\right)\sin(2nx)\,dx\right)$$

Indefinite integral

$$\int x\sin(2nx)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$$u = x, du = dx$$

$$dv = \sin(2nx)\,dx, v = \int \sin(2nx)\,dx = -\frac{1}{2n}\cos(2nx)$$

$$\int x\sin(2nx)\,dx = -\frac{1}{2n}x\cos(2nx) + \frac{1}{2n}\int \cos(2nx)\,dx =$$
$$= -\frac{1}{2n}x\cos(2nx) + \frac{1}{4n^2}\sin(2nx) + C$$

$$b_n = \frac{8}{\pi^2}\left(-\left[-\frac{1}{2n}x\cos(2nx) + \frac{1}{4n^2}\sin(2nx)\right]_{-\pi/2}^{0}\right) +$$
$$+\frac{8}{\pi^2}\left(\left[-\frac{1}{2n}x\cos(2nx) + \frac{1}{4n^2}\sin(2nx)\right]_{0}^{\pi/2}\right) =$$
$$= \frac{8}{\pi^2}\left(-\left(-0 + 0 - \left(-\frac{1}{2n}(-\pi/2)\cos(2n(-\pi/2))\right)\right)\right) +$$
$$+\frac{8}{\pi^2}\left(-\frac{1}{2n}(\pi/2)\cos(2n(\pi/2)) + 0 - (-0 + 0)\right) = 0$$

We could therefore write the series as

$$\frac{2}{2} - \frac{8}{\pi^2(2k+1)^2}\cos(2(2k+1)x)\,, k = 0, 1, 2, \ldots$$

$$1 - \frac{8}{\pi^2(2k+1)^2}\cos(2(2k+1)x)\,, k = 0, 1, 2, \ldots$$

$\boldsymbol{Answer}$: $1 - \frac{8}{\pi^2(2k+1)^2}\cos(2(2k+1)x)\,, k = 0, 1, 2, \ldots$