

## Answer on Question #78893 – Math – Differential Equations

### Question

Reduce the following PDE to a set of three ODEs by the method of separation of variables

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \left( \frac{\partial V}{\partial r} \right) \right) + \left( \frac{1}{r^2} \right) \left( \frac{\partial^2 V}{\partial \theta^2} \right) + \left( \frac{\partial^2 V}{\partial z^2} \right) + (k^2)V = 0$$

### Solution

Assume

$$V(r, \theta, z) = R(r)\Theta(\theta)Z(z)$$

$$\Theta Z \left( \frac{1}{r} \right) \frac{d}{dr} \left( r \left( \frac{dR}{dr} \right) \right) + RZ \left( \frac{1}{r^2} \right) \left( \frac{d^2 \Theta}{d\theta^2} \right) + R\Theta \left( \frac{d^2 Z}{dz^2} \right) + (k^2)R\Theta Z = 0$$

Divide through by  $V$ :

$$\frac{1}{R} \left( \frac{1}{r} \right) \frac{d}{dr} \left( r \left( \frac{dR}{dr} \right) \right) + \frac{1}{\Theta} \left( \frac{1}{r^2} \right) \left( \frac{d^2 \Theta}{d\theta^2} \right) + \frac{1}{Z} \left( \frac{d^2 Z}{dz^2} \right) + k^2 = 0$$

$$\frac{1}{R} \left( \frac{1}{r} \right) \frac{dR}{dr} + \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{\Theta} \left( \frac{1}{r^2} \right) \left( \frac{d^2 \Theta}{d\theta^2} \right) + k^2 = -\frac{1}{Z} \left( \frac{\partial^2 Z}{\partial z^2} \right)$$

In the above equation the left-hand side depends on  $r$  and  $\theta$ , while the right-hand side depends on  $z$ . The only way these two members are going to be equal for all values of  $r$ ,  $\theta$  and  $z$  is when both of them are equal to a constant. Let us define such a constant as  $-l^2$ .

With this choice for the constant, we obtain:

$$\frac{d^2 Z}{dz^2} - l^2 Z = 0$$

The general solution of this equation is:

$$Z(z) = a_1 e^{lz} + a_2 e^{-lz}$$

Such a solution, when considering the specific boundary conditions, will allow  $Z(z)$  to go to zero for  $z$  going to  $\pm\infty$ , which makes physical sense. If we had given the constant a negative value, we would have had periodic trigonometric functions, which do not tend to zero for  $z$  going to infinity.

Once sorted the  $z$ -dependency, we need to take care of  $r$  and  $\theta$ .

$$\frac{1}{R} \left( \frac{1}{r} \right) \frac{dR}{dr} + \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{\Theta} \left( \frac{1}{r^2} \right) \left( \frac{d^2 \Theta}{d\theta^2} \right) = -(k^2 + l^2)$$
$$\frac{r}{R} \frac{dR}{dr} + \frac{r^2}{R} \frac{d^2 R}{dr^2} + (k^2 + l^2)r^2 = -\frac{1}{\Theta} \left( \frac{d^2 \Theta}{d\theta^2} \right)$$

Again we are in a situation where the only way a solution can be found for the above equation is when both members are equal to a constant. This time we select a positive constant, which we call  $m^2$ . The equation for  $\Theta$  becomes, then:

$$\frac{d^2\theta}{d\theta^2} + m^2\theta = 0$$

Its general solution can be written as:

$$\theta(\theta) = b_1 \sin(m\theta) + b_2 \cos(m\theta)$$

This solution is well suited to describe the variation for an angular coordinate like  $\theta$ . Had we chosen to set both members of equation equal to a negative number, we would have ended up with exponential functions with a different value assigned to  $\theta(\theta)$  for each 360 degrees turn, a clear non-physical solution.

Last to be examined is the r-dependency. We have:

$$\begin{aligned} \frac{r}{R} \frac{dR}{dr} + \frac{r^2}{R} \frac{d^2R}{dr^2} + (k^2 + l^2)r^2 &= m^2 \\ r^2 \frac{d^2R}{dr^2} + r \frac{dR}{dr} + ((k^2 + l^2)r^2 - m^2)R &= 0 \end{aligned} \quad (*)$$

The equation (\*) is a well-known equation of mathematical physics called parametric Bessel's equation. With a simple linear transformation of variable,  $x = (\sqrt{k^2 + l^2})r$ , equation (\*) is readily changed into a Bessel's equation:

$$\begin{aligned} \frac{dR}{dr} &= \frac{dR}{dx} \frac{dx}{dr} = (\sqrt{k^2 + l^2}) R' \\ \frac{d^2R}{dr^2} &= \frac{d}{dx} \left( (\sqrt{k^2 + l^2}) R' \right) \frac{dx}{dr} = (k^2 + l^2) R'' \\ \frac{x^2}{k^2 + l^2} (k^2 + l^2) R'' + \frac{x}{\sqrt{k^2 + l^2}} (\sqrt{k^2 + l^2}) R' + (x^2 - m^2) R &= 0 \\ x^2 R'' + x R' + (x^2 - m^2) R &= 0 \end{aligned}$$

where  $R''$  and  $R'$  indicate the first and the second derivatives with respect to  $x$ .

In what follows we will assume that  $m$  is a real, non-negative number.

Linearly independent solutions are typically denoted  $J_m(x)$  (Bessel Functions) and  $N_m(x)$  (Neumann Functions).