Answer on Question #78889 – Math – Analytic Geometry

Question

Find the equation of the conic of which one focus lies at $(2,1)$ one directrix is $x + y = 0$ and it passes through $(1,4)$. Also identify the conic and reduce the conic you obtained above to standard form.
Draw a rough sketch of the conic obtained above.

Solution

Conic is defined as locus of a point moving in a plane such that the ratio of its distance from a fixed point (F) to the fixed straight line is always a constant. This ratio is called as eccentricity.
The distance of point $(1,4)$ from focus at $(2,1)$ is

$$\sqrt{(1-2)^2+(4-1)^2}=\sqrt{10}$$

The distance of point $(1,4)$ from directrix $x + y = 0$ is

$$\frac{|1+4|}{\sqrt{(1)^2+(1)^2}}=\frac{5\sqrt{2}}{2}$$

Find the eccentricity

$$e=\frac{\sqrt{10}}{\frac{5\sqrt{2}}{2}}=\frac{2\sqrt{5}}{5}<1$$

Hence we have the ellipse.
The distance from an arbitrary point (x,y) to the focus $(2,1)$

$$\sqrt{(x-2)^2+(y-1)^2}$$

The distance of the point (x,y) from directrix $x + y = 0$

$$\frac{|x+y|}{\sqrt{(1)^2+(1)^2}}=\frac{|x+y|}{\sqrt{2}}$$

The eccentricity

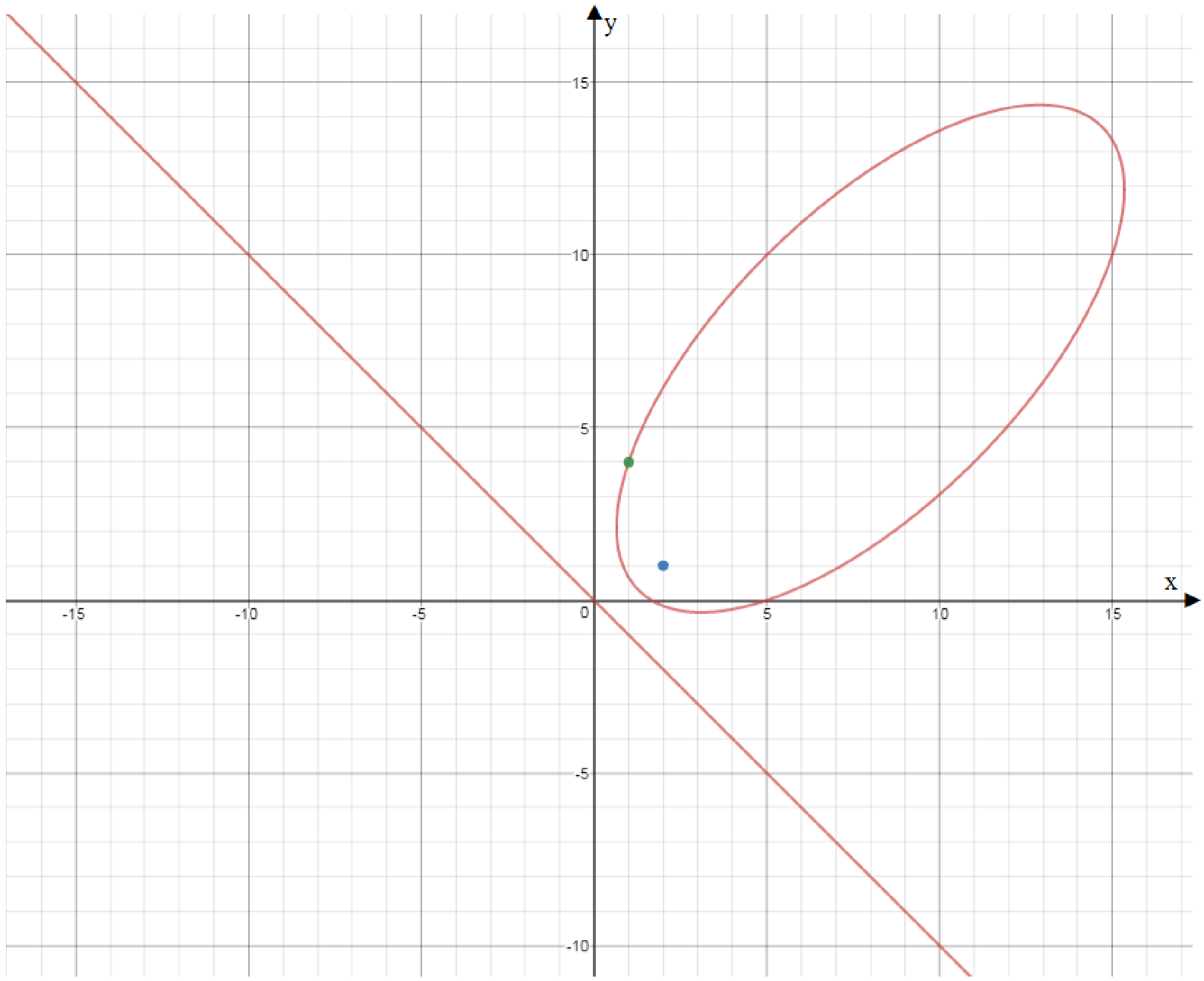
$$e=\frac{\sqrt{(x-2)^2+(y-1)^2}}{\frac{|x+y|}{\sqrt{2}}}=\frac{2\sqrt{5}}{5}$$

$$(x-2)^2+(y-1)^2=\frac{2}{5}(x+y)^2$$

$$x^2-4x+4+y^2-2y+1=\frac{2}{5}x^2+\frac{4}{5}xy+\frac{2}{5}y^2$$

$$\frac{3}{5}x^2-\frac{4}{5}xy+\frac{3}{5}y^2-4x-2y+5=0$$

$$3x^2-4xy+3y^2-20x-10y+25=0$$



$$A = 3, B = -4, C = 3, D = -20, E = -2, F = 25$$

$$\cot(2\theta) = \frac{A - C}{B} = \frac{3 - 3}{-4} = 0 => \theta = 45°$$

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta$$

$$x = x' \frac{\sqrt{2}}{2} - y' \frac{\sqrt{2}}{2}$$
$$y = x' \frac{\sqrt{2}}{2} + y' \frac{\sqrt{2}}{2}$$

$$3\left(x' \frac{\sqrt{2}}{2} - y' \frac{\sqrt{2}}{2}\right)^2 - 4\left(x' \frac{\sqrt{2}}{2} - y' \frac{\sqrt{2}}{2}\right)\left(x' \frac{\sqrt{2}}{2} + y' \frac{\sqrt{2}}{2}\right) +$$

$$+3\left(x'\frac{\sqrt{2}}{2}+y'\frac{\sqrt{2}}{2}\right)^2-20\left(x'\frac{\sqrt{2}}{2}-y'\frac{\sqrt{2}}{2}\right)-10\left(x'\frac{\sqrt{2}}{2}+y'\frac{\sqrt{2}}{2}\right)+25=0$$

$$3\left(\frac{(x')^2}{2}-x'y'+\frac{(y')^2}{2}\right)-4\left(\frac{(x')^2}{2}-\frac{(y')^2}{2}\right)+3\left(\frac{(x')^2}{2}+x'y'+\frac{(y')^2}{2}\right)-$$

$$-10\sqrt{2}x'+10\sqrt{2}y'-5\sqrt{2}x'-5\sqrt{2}y'+25=0$$

$$(x')^2-15\sqrt{2}x'+5(y')^2+5\sqrt{2}y'+25=0$$

$$(x')^2-2\left(\frac{15\sqrt{2}}{2}\right)x'+\frac{225}{2}-\frac{225}{2}+5\left((y')^2+2\left(\frac{5\sqrt{2}}{2}\right)y'+\frac{25}{2}-\frac{25}{2}\right)+25=$$

$$=0$$

$$\left(x'-\frac{15\sqrt{2}}{2}\right)^2+5\left(y'+\frac{5\sqrt{2}}{2}\right)^2=150$$

$$\frac{\left(x'-\frac{15\sqrt{2}}{2}\right)^2}{\left(5\sqrt{6}\right)^2}+\frac{\left(y'+\frac{5\sqrt{2}}{2}\right)^2}{\left(\sqrt{30}\right)^2}=1$$