Answer on Question #78658 – Math – Analytic Geometry

Question

Prove that the product of the distance from any point on a hyperbola to its asymptotes is a constant.

Solution

Let there be given a hyperbola. If the axes of a rectangular coordinate system are chosen so that the foci of the given hyperbola are symmetrically situated on the x −axis with respect to the origin, then the equation of the hyperbola has the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The equations of the asymptotes are

$$y = \frac{b}{a}x\,, \quad y = -\frac{b}{a}x$$

or

$$t_1: bx - ay = 0, \quad t_2: bx + ay = 0$$

Let $P(x_0, y_0)$ be any point on a hyperbola. Then the distance d_1 from P to t_1 is given by

$$d_1 = \frac{|bx_0 - ay_0|}{\sqrt{a^2 + b^2}}$$

The distance d_2 from P to t_2 is given by

$$d_2 = \frac{|bx_0 + ay_0|}{\sqrt{a^2 + b^2}}$$

Hence

$$d_1 d_2 = \frac{|bx_0 - ay_0|}{\sqrt{a^2 + b^2}} \cdot \frac{|bx_0 + ay_0|}{\sqrt{a^2 + b^2}} = \frac{|b^2 x_0^2 - a^2 y_0^2|}{a^2 + b^2}$$

But $P(x_0, y_0)$ is the point on the hyperbola. Then

$$\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1 => b^2 x_0^2 - a^2 y_0^2 = a^2 b^2$$

Therefore

$$d_1 d_2 = \frac{|b^2 x_0^2 - a^2 y_0^2|}{a^2 + b^2} = \frac{a^2 b^2}{a^2 + b^2} = const$$

We prove that the product of the distance from any point on a hyperbola to its asymptotes is a constant.