Answer on Question #78594 – Math – Analytic Geometry
Question

Prove that the paraboloids

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=\frac{2z}{c_1}$$

$$\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=\frac{2z}{c_2}$$

$$\frac{x^2}{a_3^2}+\frac{y^2}{b_3^2}=\frac{2z}{c_3}$$

have a common tangent plane if

$$\begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1^2 & c_2^2 & c_3^2 \end{vmatrix}=0$$

Solution

A normal vector to the surface $f(x,y,z)=c$ at $(x_0\,,y_0\,,z_0\,)$ is given by

$$\nabla f(x_0\,,y_0\,,z_0\,)$$

Then the normal vector n_i to the paraboloid P_i: $\frac{x^2}{a_i^2}+\frac{y^2}{b_i^2}-\frac{2z}{c_i}=0$ is given by

$$\overrightarrow{n_i}=\left(\frac{{x_0}^2}{a_i^2},\frac{{y_0}^2}{b_i^2},-\frac{2z_0}{c_i}\right)$$

If there exists a common tangent plane then

$$\begin{cases} \overrightarrow{n_1}=\lambda_2\overrightarrow{n_2} \\ \overrightarrow{n_1}=\lambda_3\overrightarrow{n_3} \end{cases}$$

We have that

$$\begin{cases} \frac{{x_0}^2}{a_1^2}=\lambda_2\frac{{x_0}^2}{a_2^2} \\ \frac{{y_0}^2}{b_1^2}=\lambda_2\frac{{y_0}^2}{b_2^2} \\ \frac{2z_0}{c_1}=\lambda_2\frac{2z_0}{c_2} \end{cases}$$

$$\begin{cases} \frac{{x_0}^2}{a_1^2} = \lambda_3 \frac{{x_0}^2}{a_3^2} \\ \frac{{y_0}^2}{b_1^2} = \lambda_3 \frac{{y_0}^2}{b_3^2} \\ \frac{2z_0}{c_1} = \lambda_3 \frac{2z_0}{c_3} \end{cases}$$

This means that the columns in the matrix

$$\begin{pmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1^2 & c_2^2 & c_3^2 \end{pmatrix}$$

are linearly dependent.

Therefore,

$$\begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1^2 & c_2^2 & c_3^2 \end{vmatrix} = 0$$