Answer on Question #78512 – Math – Analytic Geometry

Question

Under a rotation of axes, a parabola can become a hyperbola. Is the statement true? Give reason for your answer, either with a short proof or a counterexample.

Solution

The general equation of the second-order curve is given by

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{31}x + 2a_{23}y + a_{33} = 0,$$

where a_{ii} are some coefficients. The class of the curve (ellipse, hyperbola, parabola) is defined by the following determinant

$$\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

If $\delta > 0$ then the curve is ellipse, if $\delta = 0$, then parabola, else hyperbola.

First three members of the equation may be rewritten in the following way

$$Q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotation of axes is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix},$$

where x', y' are new coordinates.

Substituting it into $Q(x, y)$ we obtain

$$Q(x, y) = \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^T \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Thus,

$$\delta' = \det\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^T \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\} =$$

$$= \det \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^T \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \det \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = |\sin^2\theta + \cos^2\theta = 1| =$$

$$= \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \delta$$

This means that under such transformation δ doesn't change, thus the curve can't change its class. So the statement is false.

Answer: false.