ANSWER on Question #77405 – Math – Differential Equations

QUESTION

Solve the partial differential equation

$$(D^2 + 3DD' + 2D'^2)z = 12xy$$

SOLUTION

Let us use some known facts.

Let the given differential equation be

$$F(D, D') = f(x, y).$$

Factorize $F(D, D')$ into linear factors. Then use the following results:

Rule I. Corresponding to each non-repeated factor $(bD - aD' - c)$, the part of C.F. is taken as

$$e^{\left(\frac{cx}{b}\right)}\varphi(by + ax), \;\; if\ b \neq 0$$

We now have three particular cases of Rule I:

Rule IA. Take $c = 0$ in Rule I. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is

$$\varphi(by + ax), \;\; if\ b \neq 0.$$

Rule IB. Take $a = 0$ in Rule I. Hence corresponding to each linear factor $(bD - c)$, the part of C.F. is

$$e^{\left(\frac{cx}{b}\right)}\varphi(by), \;\; if\ b \neq 0.$$

Rule IC. Take $a = c = 0$ and $b = 1$ in Rule I. Hence corresponding to each linear factor $(1 \cdot D)$, the part of C.F. is

$$\varphi(y).$$

In our case,

$$(D^2 + 3DD' + 2D'^2)z = 12xy \to z(x,y) = C.F. + P.I.$$

0 STEP: We factor the expression

$$D^2 + 3DD' + 2D'^2 = (D^2 + DD') + (2DD' + 2D'^2) =$$

$$= D(D + D') + 2D'(D + D') = (D + D')(D + 2D')$$

Conclusion,

$$\boxed{(D^2 + 3DD' + 2D'^2)z = (D + D')(D + 2D')z}$$

1 STEP: Let find C.F.

$$\begin{cases} (D + D')z \\ (bD - aD' - c)z \end{cases} \to \begin{cases} b = 1 \\ a = -1 \\ c = 0 \end{cases} \to (C.F.)_1 = e^{\left(\frac{0 \cdot x}{1}\right)} \cdot \varphi_1(1 \cdot y + (-1) \cdot x) \to$$

$$\boxed{(C.F.)_1 = \varphi_1(y - x), where\ \varphi_1\ is\ arbitrary\ function}$$

$$\begin{cases} (D + 2D')z \\ (bD - aD' - c)z \end{cases} \to \begin{cases} b = 1 \\ a = -2 \\ c = 0 \end{cases} \to (C.F.)_2 = e^{\left(\frac{0 \cdot x}{1}\right)} \cdot \varphi_2(1 \cdot y + (-2) \cdot x) \to$$

$$\boxed{(C.F.)_2 = \varphi_2(y - 2x), where\ \varphi_2\ is\ arbitrary\ function}$$

Then,

$$C.F. = (C.F.)_1 + (C.F.)_2 \to \boxed{C.F. = \varphi_1(y - x) + \varphi_2(y - 2x)}$$

2 STEP: Let find P.I.

$$P.I. = \frac{1}{(D + D')(D + 2D')}(12xy) = \frac{1}{D\left(1 + \frac{D'}{D}\right)D\left(1 + \frac{2D'}{D}\right)}(12xy) =$$

$$=\frac{1}{D^2}\cdot\left[1+\frac{D'}{D}\right]^{-1}\left[1+\frac{2D'}{D}\right]^{-1}(12xy)=\left[\frac{1}{1+x}=1-x+x^2-x^3+x^4+\cdots,|x|<1\right]=$$

$$=\frac{1}{D^2}\cdot\left[1-\frac{D'}{D}+\cdots\right]\left[1-\frac{2D'}{D}+\cdots\right](12xy)=$$

$$=\frac{1}{D^2}\cdot\left[1-\frac{D'}{D}-\frac{2D'}{D}+\frac{2D'^2}{D^2}\dots\right](12xy)=\frac{1}{D^2}\cdot\left[1-\frac{3D'}{D}+\frac{2D'^2}{D^2}\dots\right](12xy)=$$

$$=\frac{1}{D^2}\cdot\left[12xy-\frac{3}{D}\cdot\left(\frac{\partial}{\partial y}(12xy)\right)+\frac{2}{D^2}\cdot\left(\frac{\partial^2}{\partial y^2}(12xy)\right)+\cdots\right]=$$

$$=\left[\frac{1}{D}\equiv\int dx\right]=\frac{1}{D^2}\cdot\left[12xy-\frac{3}{D}\cdot 12x+\frac{2}{D^2}\cdot 0+\cdots\right]=\frac{1}{D^2}\cdot\left[12xy-(3\cdot 12)\cdot\int xdx\right]=$$

$$=\frac{1}{D^2}\cdot\left[12xy-36\cdot\frac{x^2}{2}\right]=\frac{1}{D}\cdot\left(\int(12xy-18x^2)\,dx\right)=\frac{1}{D}\cdot\left(\frac{12yx^2}{2}-\frac{18x^3}{3}\right)=$$

$$=\frac{1}{D}\cdot(6yx^2-6x^3)=\int(6yx^2-6x^3)dx=\frac{6yx^3}{3}-\frac{6x^4}{4}=2yx^3-\frac{3x^4}{2}$$

Then,

$$\boxed{P.I.=2yx^3-\frac{3x^4}{2}}$$

Conclusion,

$$z(x,y)=C.F.+P.I.=\varphi_1(y-x)+\varphi_2(y-2x)+2yx^3-\frac{3x^4}{2}$$

$$\boxed{\begin{cases}z(x,y)=\varphi_1(y-x)+\varphi_2(y-2x)+2yx^3-\dfrac{3x^4}{2}\\ \quad where\ \varphi_1 and\ \varphi_2\ are\ arbitrary\ functions\end{cases}}$$

ANSWER: $\begin{cases}z(x,y)=\varphi_1(y-x)+\varphi_2(y-2x)+2yx^3-\frac{3x^4}{2}\\ \quad where\ \varphi_1 and\ \varphi_2\ are\ arbitrary\ functions\end{cases}$