Answer on Question #76519 – Math – Algebra

Question

1) Find the length of an arc of the curve given
by x = a(theta — sin theta), y = a(1— cos theta), and

2) obtain the ratio in which theta = π/3 divides it.

Solution

We have

$$\begin{cases} x = a(\theta - sin\theta) \\ y = a(1 - cos\theta) \end{cases}$$

1) If we draw the plot of this curve, we see that the arc has the origin in the point θ =0 and the end in the point θ =2π.

We will find the length with formula:

$$\mathrm{L} = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = [a(\theta - sin\theta)]' = a'(\theta - sin\theta) + a[\theta' - (sin\theta)'] = 0 + a(1 - cos\theta) = a(1 - cos\theta)$$

$$\frac{dy}{d\theta} = [a(1 - cos\theta)]' = a'(1 - cos\theta) + a[1' - (cos\theta)'] = asin\theta$$

$$\mathrm{L} = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta, \text{ where } \theta_1 = 0 \text{ and } \theta_2 = 2\pi$$

$$\mathrm{L} = \int_0^{2\pi} \sqrt{a^2(1 - \cos\theta)^2 + a^2 sin\theta^2}\ d\theta = a\int_0^{2\pi} \sqrt{1 - 2cos\theta + cos^2\theta + sin^2\theta}\ d\theta =$$

$$= a\int_0^{2\pi} \sqrt{2 - 2cos\theta}\, d\theta = a\int_0^{2\pi} \sqrt{2(1 - cos\theta)}\, d\theta = a\int_0^{2\pi} \sqrt{2\left(2sin^2\frac{\theta}{2}\right)}\, d\theta =$$

$$= 2a\int_0^{2\pi} sin\frac{\theta}{2} = 4a\left(-cos\frac{\theta}{2}\Big|_0^{2\pi}\right) = 4a(-cos\pi + cos0) = 4a(1 + 1) = 8a$$

So, L = 8a;

2) $If\ \theta = \frac{\pi}{3}$, the ratio in which θ divides the curve looks like:

$$\frac{4a\left(-cos\frac{\theta}{2}\Big|_0^{\frac{\pi}{3}}\right)}{4a\left(-cos\frac{\theta}{2}\Big|_{\frac{\pi}{3}}^{2\pi}\right)} = \frac{4a(-cos\frac{\pi}{6}+cos0)}{4a\left(-cos\pi+cos\frac{\pi}{6}\right)} = \frac{-\frac{\sqrt{3}}{2}+1}{1+\frac{\sqrt{3}}{2}} = \frac{2-\sqrt{3}}{2+\sqrt{3}}$$

Answer: 1) 8a; 2) $\frac{2-\sqrt{3}}{2+\sqrt{3}}$.