Answer on Question #76448, Math / Calculus

Let the function $f: R^2 \to R$ is defined as follow

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & if\ (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Then show that

1) $f_x(0,y) = -y$ for all y

Solution

$(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^3y - xy^3}{x^2+y^2}\right) = \frac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} =$$

$$= \frac{3x^4y - x^2y^3 + 3x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2} =$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_x(0,y) = \frac{0 + 0 - y^5}{(0+y^2)^2} = -y$$

$(x,y) = (0,0)$

$$f_x(0,0) = \frac{\partial}{\partial x}\big(f(0,0)\big) = \lim_{h\to 0}\frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

Therefore,

$f_x(0,y) = -y$ for all y

2) $f_y(x,0) = x$ for all x

Solution

$(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^3y - xy^3}{x^2+y^2}\right) = \frac{(x^3 - 3xy^2)(x^2+y^2) - 2y(x^3y - xy^3)}{(x^2+y^2)^2} =$$

$$= \frac{x^5 + x^3y^2 - 3x^3y^2 - 3xy^4 - 2x^3y^2 + 2xy^4}{(x^2+y^2)^2} =$$

$$= \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

$$f_y(x,0) = \frac{x^5 - 0 - 0}{(x^2+0)^2} = x$$

$(x,y) = (0,0)$

$$f_y(0,0) = \frac{\partial}{\partial x}\big(f(0,0)\big) = \lim_{h\to 0}\frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

Therefore,

$f_y(x,0) = x$ for all x

3) So that $f_{xy}(0,0) \neq f_{yx}(0,0)$

Solution

$$f_{xy}(0,0) = \lim_{u \to 0} \frac{f_x(0, 0+u) - f_x(0,0)}{u} = \lim_{u \to 0} \frac{-u - 0}{u} = -1$$

$$f_{yx}(0,0) = \lim_{u \to 0} \frac{f_y(0+u, 0) - f_y(0,0)}{u} = \lim_{u \to 0} \frac{u - 0}{u} = 1$$

So that $f_{xy}(0,0) \neq f_{yx}(0,0)$